FACTORIZATION HOMOLOGY I: HIGHER CATEGORIES

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Abstract. We construct a pairing, which we call factorization homology, between framed manifolds and higher categories. The essential geometric notion is that of a vari-framing of a stratified manifold, which is a framing on each stratum together with a coherent system of compatibilities of framings along links of strata. Our main result constructs labeling systems on disk-stratified vari-framed n-manifolds from (∞,n)-categories. These (∞,n)-categories, in contrast with the literature to date, are not required to have adjoints. The core calculation supporting this result is the contractibility of the space of conically smooth diffeomorphisms of a disk-stratified manifold which preserve a vari-framing. This allows the following conceptual definition: the factorization homology

$$\int_M \mathcal{C}$$

of a framed n-manifold M with coefficients in an (∞,n)-category $\mathcal{C}$ is the classifying space of $\mathcal{C}$-labeled disk-stratifications over M.

Contents

2010 *Mathematics Subject Classification.* Primary 58D29. Secondary 57R56, 57N80, 57S05, 57R19, 18B30, 57R15.
Key words and phrases. Factorization homology. Stratified spaces. Vari-framed stratified manifolds. (∞,n)-Categories. Complete Segal spaces. Quasi-categories. Exit-path categories. Striation sheaves.
DA was supported by the National Science Foundation under award 1507704. JF was supported by the National Science Foundation under awards 1207758 and 1508040. Parts of this paper were written while JF was a visitor at the Mathematical Sciences Research Institute and at Université Pierre et Marie Curie.

Introduction

In this work, we construct the factorization homology of n-manifolds with coefficients in (∞, n)-categories. We posit this forms the fundamental relation between manifold topology and higher category theory, answering a question which we now motivate and describe.

In 1988, Atiyah [At] proposed a mathematical framework for topological quantum field theory modeled on Segal's earlier axioms for conformal field theory [Se2]. An explosion in physically motivated topology over the previous five years informed his proposal. These advances were carried out by new studies of gauge theory; this includes both Atiyah & Bott's analysis of the Morse theory of the Yang–Mills functional to compute the cohomology of algebraic bundles on Riemann surfaces in [AB], as well as Donaldson's revolution in smooth 4-manifold topology based on the self-dual Yang–Mills equations in [Do]. These advances led to an open challenge to marry other low-dimensional topology invariants, such as the Casson invariant and the Jones polynomial, with mathematical physics. Witten answered this challenge by introducing Chern–Simons theory [Wi], a gauge theory in which the standard Yang–Mills action is replaced by the Chern–Simons 3-form of the connection. At a physical level of rigor, Witten showed that the Jones polynomial is the partition function in Chern–Simons theory.

Atiyah's proposed axioms were most influenced by Chern–Simons theory and Witten's notion of topologically invariant quantum field theories. In Chern–Simons, a 3-manifold M is assigned an element in a vector space $Z(\partial M)$ associated to its boundary. This association adheres to a locality with respect to surgery on manifolds. Atiyah added axioms to encode this surgery-locality in terms of Thom's cobordism theory: in this now ubiquitous definition, a topological quantum field theory is a functor from a category whose object are $(n-1)$-manifolds and whose morphisms are n-dimensional cobordisms.

By the early 1990s, it had become clear that if codimension-1 boundary conditions form a vector space, then higher codimension defects should correspond to higher categorical objects. Earliest publications of this include works of Lawrence [La], Freed [Fr1], and Crane–Yetter [CY], but the insight is often attributed collectively to many mathematicians, including Baez, Dolan, Kapranov, Kazhdan, Reshetikhin, Turaev, Voevodsky, and others. Relevant works include [BaDo], [KV], [RT], [Wa], and [Ka]; see in particular, Freed's work on quantum groups [Fr2] and the Baez–Dolan cobordism hypothesis, which specified many features that should be true of this connection between manifolds and higher category theory in terms of an extensive surgery-locality based on Morse theory.

While it appeared clear that higher categories bore a close connection to field theory, a basic question remained unanswered: what is it that connects them? For instance, field theories are defined by integration – is there integration on the categorical side? Or does the category theory only serve as an elaborate system of bookkeeping?

During this same period, Beilinson & Drinfeld introduced a beautiful theory of chiral and factorization algebras, an algebro-geometric approach to conformal field theory; their work was finally published a decade later in [BeDr]. Therein, they devised a fantastic procedure—chiral homology—in which one integrates a chiral algebra coherently over all configuration spaces of a curve to produce a conformal field theory. The conformal blocks of the field theory occur as the zeroth chiral homology group. They defined algebro-geometric forms of standard vertex algebras, and calculated their chiral homologies in several cases of especial interest, including lattice algebras and central extensions of enveloping algebras of Lie algebras.

This theory of factorization algebras, and of coherently integrating over all configuration spaces at once, inspired and connected with a number of works in differing areas. These include: quantum groups in [BFS]; manifold topology and mapping spaces in [Lu2], [Sa], [Se3], and [AF1]; ℓ-adic cohomology and bundles on curves in [GL]. In mathematical physics, Costello [Co2] developed a rigorous system of renormalization for perturbative quantum field theories based on the Batalin–Vilkovisky formalism [BV]. Analyzed in great depth by Costello & Gwilliam [CG], the quantum observables in these renormalized theories obtain the structure of a factorization algebra in a topological sense. Assuming the theory is perturbative, then the global observables are computed by a likewise process of factorization homology: one integrates over all embedded disks, rather than configuration spaces. This theory accommodates a wealth of examples, from perturbative Chern–Simons to twisted supersymmetric gauge theories [Co3].

Consequently, for conformal field theory as well as for perturbative quantum field theory, our basic motivating question has an answer: there is integration on the categorical side, and it is chiral/factorization homology. The field theory itself is implemented by integration over manifolds from an algebraic input, which is a chiral/factorization/$\mathcal{E}_n$-algebra. However, this forms only a partial solution to our basic question, because Chern–Simons and the other field theories involved are not perturbative. Their perturbative sectors do not account for the entire theories. Said differently, the global observables in these theories are not computed as the factorization homology of the local observables, viewed as an $\mathcal{E}_n$-algebra. From the point of view of the cobordism hypothesis of Baez–Dolan, further developed by Lurie and Hopkins–Lurie in [Lu3] after Costello [Co1], certain higher categories are given by the Morita theory of $\mathcal{E}_n$-algebras. These account for those TQFTs whose value on a point is Morita equivalent to an $\mathcal{E}_n$-algebra, i.e., to an (∞, n)-category with a single object and a single k-morphism for $k < n$. (The collection of n-morphisms then forms an $\mathcal{E}_n$-algebra, just as the collection of 1-morphisms in a category with a single object forms an algebra.) For this special class of (∞, n)-categories, the outputted field theory, as expected by the cobordism hypothesis, can be implemented by taking factorization homology of $\mathcal{E}_n$-algebras. Consequently, we can now give a more precise rephrasing of our basic question.

Question 0.1. What higher codimensional enhancement of chiral/factorization homology implements topological quantum field theory?

That is, we wish to solve the theoretical problem of comparing category theory and field theory, after Baez–Dolan, within the philosophy of Beilinson–Drinfeld. In the narrative we pursue in this introduction, this theory should fill the last entry in the following table.

Physics	**Algebra**	**Integration**
CFT	chiral algebra	chiral homology ([BeDr])
perturbative TQFT	$\mathcal{E}_n$-algebra/stack	factorization homology ([Lu2], [AF1])
perturbative QFT	factorization algebra	factorization homology ([CG])
TQFT	(∞, n)-category	

Our proposed solution, which we again call factorization homology, has a simple summary: rather than integrating over configuration spaces – i.e., over the moduli space of finite subsets – integrate over a moduli space of disk-stratifications. The conclusion of this paper is that this heuristic definition can be made well-defined.

Before describing what technical features this problem absorbs and how they are overcome, we first make an observation and comment. In the diagram above, we have listed (∞, n)-categories instead of (∞, n)-categories with duals or adjoints. As far as we are aware, the TQFT literature to date has uniformly emphasized the necessity of adjoints in the category theory; these adjoints mirror categorically the Morse theory and surgery-locality of Atiyah's axioms and the cobordism hypothesis after [BaDo] and [Lu3]. However, examples such as Donaldson theory have not fit into these axioms. There are genuine topological obstructions to defining the requisite Floer theory on the full bordism category; see [FKNSWW]. In particular, the monopole Floer homology of Kronheimer–Mrowka [KM] is defined only on a bordism category whose morphisms are connected bordisms. We are hopeful that these important Floer theories may still fit in the factorization paradigm after Beilinson & Drinfeld, exactly because we can fill in the missing square in the above diagram without requiring adjoints in the coefficient (∞, n)-categories.

We now describe our solution. First, we recall the corresponding simpler case in codimension-0, factorization homology with coefficients in an $\mathcal{E}_n$-algebra. If A is an $\mathcal{E}_n$-algebra and M is an framed n-manifold, one heuristically constructs factorization homology as

$$\int_M A \;\approx\; \Big| A\text{-labeled } n\text{-disks in } M \Big| \;,$$

the classifying space of a category, an object of which is a collection of disjointly embedded n-disks in M each of which is labeled by a point of A. There are several important classes of morphisms.

(1) **compositions**: two disks are embedded in a third disk, and the labels multiply in A.
(2) **units**: a disk is added to a configuration, labeled by the unit of A.
(3) **coherence**: disks are moved through an isotopy of embedding.

If one defines an $\mathcal{E}_n$-algebra in terms of the little n-cubes operad, using rectilinear embeddings of disks, then one has to do a calculation to show an $\mathcal{E}_n$-algebra adequately defines such a coherent labeling system on a framed n-manifold M. This calculation is that the space of rectilinear self-embeddings of a disk is homotopy equivalent to the space of framed self-embeddings of a disk. This boils down to the even more basic fact that differentiation defines a homotopy equivalence

$$\mathsf{Diff}(\mathbb{R}^n) \xrightarrow{\sim} \mathsf{fr}(\mathbb{R}^n)$$

between the space of diffeomorphisms of $\mathbb{R}^n$ and the space of framings of $\mathbb{R}^n$. Equivalently, this can be phrased as the contractibility

$$\mathsf{Diff}^{\mathsf{fr}}(\mathbb{R}^n) \simeq *$$

of the space of diffeomorphisms which preserve the framing.

We wish to make a corresponding construction where the $\mathcal{E}_n$-algebra A is replaced by an (∞,n)-category $\mathcal{C}$. The factorization homology of a framed n-manifold M with coefficients in $\mathcal{C}$ should be

$$\int_M \mathcal{C} \;\approx\; \Big|\mathcal{C}\text{-labeled disk-stratifications of } M\Big| ~,$$

the classifying space of a category, an object of consists of a coherent system of:

- a stratification of M, each closed component of which is a k-disk;
- a k-morphism of $\mathcal{C}$ for k-dimensional component of the stratification of M.

There are several important classes of morphisms.

(1) **refinements/compositions**: a stratum is refined away, forgotten, and the labels are composed.
(2) **creations/units**: a new stratum is created, labeled by identity morphisms.
(3) **coherence**: a stratification is moved via diffeomorphism to another stratification.

This template for making factorization homology is, however, afflicted by the absence of any known model for (∞,n)-categories which can define such a system of labels. Most models for (∞,n)-categories are constructed in terms of presheaves on a combinatorially defined category, such as $\mathbf{\Theta}_n$ or the n-fold product $\mathbf{\Delta}^n$, and none of these are manifestly suitable for decorating a disk-stratification.

We encountered a similar, easier, obstruction in the guiding simpler case above. One cannot obviously define factorization homology with coefficients in an algebra for the little n-cubes operad. One requires an intermediate notion, namely the operad of framed embeddings (which is infinite dimensional but homotopy finite dimensional) and a comparison result, that the two operads are weakly homotopy equivalent. This allows one to Kan extend the $\mathcal{E}_n$-algebra along the inclusion of rectilinear embeddings into framed embedding without altering the homotopy type; once one has expressed the $\mathcal{E}_n$-algebra in terms of framed embeddings, the definition is manifestly well-defined.

We solve this issue in our setting in three steps. In the first step, we construct an ∞-category of labeling systems for stratifications on framed n-manifolds. In the second step, we show that (∞,n)-categories embeds fully faithfully into labeling systems. In the third step, we define factorization homology with coefficients in the specified labeling systems. We elaborate on these steps below.

First step: In our antecedent work on striation sheaves [AFR], we constructed an ∞-category $\mathsf{c}\mathcal{B}\mathsf{un}$ whose objects are compact conically smooth stratified spaces and whose morphisms include refinements and stratum-creating maps, exactly as in points (1) and (2) above. Now, starting from $\mathsf{c}\mathcal{B}\mathsf{un}$, we restrict to the ∞-subcategory $\mathsf{c}\mathcal{D}\mathsf{isk} \subset \mathsf{c}\mathcal{B}\mathsf{un}$ of objects which are disk-stratified, as above. We then introduce the notion of a variform framing – for short, vari-framing – on a stratified space. A vari-framing consists of a framing on each stratum together with compatibilities between these framings in links of strata. From this, we define $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ as the collection of compact disk-stratified manifolds of dimension less or equal to n and equipped with a vari-framing. Lastly, the ∞-category of labeling systems is

$$\mathsf{Fun}(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces}) ~,$$

space-valued functors on vari-framed compact disk-stratified n-manifolds.

Second step: We use Rezk's presentation [Re2] of the ∞-category of (∞,n)-categories $\mathsf{Cat}_{(\infty,n)}$ as a full ∞-subcategory of $\mathsf{PShv}(\mathbf{\Theta}_n)$, presheaves on Joyal's category $\mathbf{\Theta}_n$ of [Jo2]. We construct a cellular realization

$$\mathbf{\Theta}_n^{\mathsf{op}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$$

from Joyal's category. We prove that this is fully faithful, which is the essential technical result of this paper. The core calculation underlying this fully faithfulness is the contractibility

$$\mathsf{Diff}^{\mathsf{vfr}}(\mathbb{D}^k) \;\simeq\; *$$

of the space of conically smooth diffeomorphisms of a hemispherical k-disk preserve the vari-framing. The space $\mathsf{Diff}(\mathbb{D}^k)$ is a manner of pseudoisotopy space, so this result can be interpreted as a cancellation between pseudoisotopies and vari-framings.

Third step: Lastly, we left Kan extend from $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ to $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$. That is, factorization homology is the composite

$$\int : \mathsf{Cat}_{(\infty,n)} \longrightarrow \mathsf{Fun}(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces}) \longrightarrow \mathsf{Fun}(\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces})$$

where the first functor is the fully faithful embedding of the second step, and the second functor is left Kan extension along the inclusion $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \subset \mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$. Equivalently, the factorization homology

$$\int_M \mathcal{C}$$

is the classifying space of the Grothendieck construction of the composite functor

$$\mathsf{c}\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{vfr}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \xrightarrow{\mathcal{C}} \mathcal{S}\mathsf{paces}$$

where the functor $\mathcal{C}$ is the right Kan extension of $\mathcal{C} : \mathbf{\Theta}_n^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ along the cellular realization functor $\mathbf{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$.

We now state the main result of the present work.

Theorem 0.2. *There is a fully faithful embedding of (∞, n)-categories into space-valued functors of vari-framed n-manifolds*

$$\int : \mathsf{Cat}_{(\infty,n)} \hookrightarrow \mathsf{Fun}\big(\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces}\big)$$

in which the value $\int_{\mathbb{D}^k} \mathcal{C}$ is the space of k-morphisms in $\mathcal{C}$, where $\mathbb{D}^k$ is the hemispherical k-disk.

In future works, we apply this higher codimension form of factorization homology to construct topological quantum field theories.

Future works. This work is the third paper in a larger program, currently in progress. We now outline a part of this program, in order of logical dependency. This part consists of a number of papers, the last of which proves the cobordism hypothesis, after Baez–Dolan [BaDo], Costello [Co1], Hopkins–Lurie (unpublished), and Lurie [Lu3].

[AFT]**: Local structures on stratified spaces**, by the first two authors with Hiro Lee Tanaka, establishes a theory of stratified spaces based on the notion of conical smoothness. This theory is tailored for the present program, and intended neither to supplant or even address outstanding theories of stratified spaces. This theory of conically smooth stratified spaces and their moduli is closed under the basic operations of taking products, open cones of compact objects, restricting to open subspaces, and forming open covers, and it enjoys a notion of derivative which, in particular, gives the following:

For the open cone $\mathsf{C}(L)$ on a compact stratified space L, taking the derivative at the cone-point implements a homotopy equivalence between *spaces* of conically smooth automorphisms

$$\mathsf{Aut}\big(\mathsf{C}(L)\big) \ \simeq \ \mathsf{Aut}(L) \ .$$

This work also introduces the notion of a constructible bundle, along with other classes of maps between stratified spaces.

[AFR]**: A stratified homotopy hypothesis** proves stratified spaces are parametrizing objects for ∞-categories. Specifically, we construct a functor $\mathsf{Exit}\colon \mathsf{Strat} \to \mathsf{Cat}_\infty$ and show that the resulting restricted Yoneda functor $\mathsf{Cat}_\infty \to \mathsf{PShv}(\mathsf{Strat})$ is fully faithful. The image is characterized by specific geometric descent conditions. We call these presheaves *striation sheaves*. We develop this theory so as to construct particular examples of ∞-categories by hand from stratified geometry: $\mathcal{B}\mathsf{un}$, $\mathcal{E}\mathsf{xit}$, and variations thereof. As striation sheaves, $\mathcal{B}\mathsf{un}$

classifies constructible bundles, $\mathcal{B}\mathsf{un}\colon K \mapsto \{X \xrightarrow{\mathsf{cbl}} K\}$, while $\boldsymbol{\mathcal{E}}\mathsf{xit}$ classifies constructible bundles with a section.

Present: In the present work, we construct a tangent classifier $\mathsf{T}\colon \boldsymbol{\mathcal{E}}\mathsf{xit} \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ to an ∞-category of vector spaces and injections thereamong. We use this to define ∞-categories $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ of *vari-framed compact n-manifolds*, and $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{sfr}}$ of *solidly framed compact n-manifolds.* As a striation sheaf, $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ classifies proper constructible bundles equipped with a trivialization of their fiberwise tangent classifier, and $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{sfr}}$ classifies proper constructible bundles equipped with an injection of their fiberwise tangent classifier into a trivial n-dimensional vector bundle. We then construct a functor $\mathfrak{C}\colon (\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}})^{\mathsf{op}} \to \mathsf{Cat}_{(\infty,n)}$ between ∞-categories, and use this to define *factorization homology.* This takes the form of a functor between ∞-categories

$$\int\colon \mathsf{Cat}_{(\infty,n)} \longrightarrow \mathsf{Fun}(\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces})$$

that we show is fully faithful. In this sense, vari-framed compact n-manifolds define parametrizing spaces for (∞,n)-categories. Subsequent papers characterize the essential image of this functor, and establish likewise results for (∞,n)-categories with adjoints, as they relate to solidly n-framed compact manifolds.

[AF3]**:** **The cobordism hypothesis**, by the first two authors, proves the cobordism hypothesis. Namely, for $\mathcal{X}$ a symmetric monoidal (∞,n)-category with adjoints and with duals, the space of fully extended (framed) topological quantum field theories is equivalent to the underlying ∞-groupoid of $\mathcal{X}$:

$$\mathsf{Map}^{\otimes}(\mathcal{B}\mathsf{ord}_n^{\mathsf{fr}}, \mathcal{X}) \ \simeq\ \mathcal{X}^{\sim}\ .$$

The cobordism hypothesis is a limiting consequence of the tangle hypothesis, one form of which states that, for $\ast \xrightarrow{\mathbb{1}} \mathcal{C}$ a pointed $(\infty,n+k)$-category with adjoints, there is a canonical identification $\mathsf{Map}^{\ast/}(\mathcal{T}\mathsf{ang}^{\mathsf{fr}}_{n\subset n+k}, \mathcal{C}) \simeq k\mathsf{End}_{\mathcal{C}}(\mathbb{1})$ between the space of pointed functors and the space of k-endomorphisms of the point in $\mathcal{C}$. The tangle hypothesis is proved in two steps. The first step establishes versions of the factorization homology functors above in where the higher categories are replaces by *pointed* higher categories, and the manifolds are replaced by possibly non-compact manifolds. The second step shows that the pointed $(\infty,n+k)$-category $\mathcal{T}\mathsf{ang}^{\mathsf{fr}}_{n\subset n+k}$, as a copresheaf on $\mathcal{M}\mathsf{fd}^{\mathsf{sfr}}_{n+k}$, is represented by the object $\mathbb{R}^k$. The bordism hypothesis follows from the tangle hypothesis, represented by the equivalence $\mathcal{B}\mathsf{ord}_n^{\mathsf{fr}} \simeq \varinjlim \Omega^k\mathbb{R}^k$ as copresheaves on $\mathcal{M}\mathsf{fd}_n^{\mathsf{sfr}}$.

Linear overview. We conclude the introduction by a linear overview of this work, followed by a comparison with spiritually similar works.

Section 1 recalls the requisite definitions and results on stratified spaces from the antecedent works [AFR] and [AFT]. In the joint work [AFT] with Hiro Lee Tanaka, a theory of smoothly stratified spaces founded on the key technical notion of conical smoothness was developed. This technical feature allowed for well-behaved homotopy types of mapping spaces and such bedrock results as an inverse function theorem, an isotopy extension theorem, and the unzipping construction, which is a functorial resolution of singularities. One could take the collective results of [AFT] as meaning that there is a theory of stratified spaces with well-behaved smooth moduli.

In [AFR], we developed this theory further, showing that this theory extends to one with well-behaved singular moduli. An ∞-category $\mathcal{B}\mathsf{un}$ encodes this theory of singular moduli of stratified spaces. A morphism in $\mathcal{B}\mathsf{un}$ can be regarded as a constructible bundle over the standardly stratified interval $\{0\} \subset [0,1]$. A close relative, the absolute exit-path ∞-category $\boldsymbol{\mathcal{E}}\mathsf{xit}$, likewise encodes the theory of pointed singular moduli. The construction of $\mathcal{B}\mathsf{un}$ and $\boldsymbol{\mathcal{E}}\mathsf{xit}$ formed the main result of that work. Their existence is nonformal, because 1-morphisms do not obviously compose: one wants to compose by gluing intervals end-to-end, but the resulting total space no longer maps constructibly to interval. One must resolve singularities and retract floating strata to fix the total space.

Consequently, the existence of $\mathcal{B}\mathsf{un}$ and $\mathcal{E}\mathsf{xit}$ as ∞-categories requires one to verify horn-filling conditions by hand using a sort of dévissage of conically smooth stratified structures. In order to perform this by-hand construction, we broke the problem into two conceptual steps. First, we introduced striation sheaves, sheaves on stratified spaces which satisfy additional descent conditions, and we proved that $\mathcal{B}\mathsf{un}$ and $\mathcal{E}\mathsf{xit}$ are striation sheaves. This required showing this theory of singular moduli satisfied descent for blow-ups and for gluing along consecutive strata, among other conditions. Second, we proved that striation sheaves are equivalent to ∞-categories. To do so, we showed that there is a fully faithful functor

$$\mathcal{S}\mathsf{trat} \xrightarrow{\mathsf{Exit}} \mathsf{Cat}_\infty$$

given by a complete Segal space form of the exit-path ∞-category of Lurie [Lu2] and MacPherson. Our construction of Exit is by restricting the Yoneda embedding along a functor

$$\mathsf{st} : \mathbf{\Delta} \hookrightarrow \mathcal{S}\mathsf{trat}$$

defined by sends the object $[p]$ to the topological p-simplex Δ^p with the standard stratification – the conical stratification given by regarding Δ^p as the p-fold cone on a point. By analysis of the homotopy type of conically smooth stratified maps between cones, we obtained that the functor st is fully faithful, and thus that there is an embedding of $\mathsf{PShv}(\mathbf{\Delta})$, hence ∞-categories, into presheaves on stratified spaces. The result then followed by applying a dévissage of stratified spaces, showing that the values of a striation sheaf are determined by two values (on a point and on a 1-simplex) after a combination of resolving singularities and induction on depth of singularity type.

Section 2 begins the new material of the present work. Intuitively, presheaves on the ∞-category $\mathsf{c}\mathcal{B}\mathsf{un}$ should present a theory of "(∞,∞)-categories with pseudoisotopic duals" in the same way that presheaves on the simplex category $\mathbf{\Delta}$ presents usual ∞-categories. However, in this work we do not want to express "(∞,∞)-categories with pseudoisotopic duals" in terms of manifolds; our goal is merely to express (∞,n)-categories in terms of manifolds. Consequently, we modify $\mathsf{c}\mathcal{B}\mathsf{un}$ in two ways for this purpose.

(1) We restrict to objects of $\mathsf{c}\mathcal{B}\mathsf{un}$ of dimension less than or equal to n. This has the effect of eliminating noninvertible k-morphisms for $k > n$.
(2) We introduce a stratified tangential structure – a vari-framing. This has the effect of eliminating the duals and pseudoisotopies.

In order to define the vari-framing, in §2.3 we enhance the usual definition of the tangent bundle of a stratified space (see §2.1 of [Pf]) so as to be defined in singular families parametrized by stratified spaces. A vari-framing of a single stratified space is a trivialization of its constructible tangent bundle. This is a more subtle notion than just a framing on the underlying space. A vari-framing of a family is then a trivialization of the vertical constructible tangent bundle. We then define $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ as the sheaf on stratified spaces which classifies such proper families of vari-framed n-manifolds. To verify that $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ satisfies the striation sheaf axioms, and so defines an ∞-category, we reduce to showing that the functor $\mathcal{E}\mathsf{xit} \to \mathcal{B}\mathsf{un}$ satisfies the exponentiability property of Conduché – this is shown is shown in the appendix.

Section 3 equates the topology of the constructions of §2 with combinatorics. First, we cut down the topology by restricting to $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$, a full ∞-subcategory of $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ whose objects are disk-stratified. This notion is defined in §3.3. The principal construction of this section is a cellular realization functor

$$\mathbf{\Theta}_n^{\mathsf{op}} \xrightarrow{\langle - \rangle} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$$

from Joyal's category $\mathbf{\Theta}_n$, the n-fold wreath product of $\mathbf{\Delta}$, the usual simplex category. We construct our cellular realization in two steps.

- For $n = 1$, we directly construct the cellular realization $\mathbf{\Theta}_1^{\mathsf{op}} = \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}$.
- We construct a functor $\left(\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}\right)^{\wr n} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$.

Lastly, we prove that this cellular realization is fully faithful, which is the main technical result of this paper. After an analysis of colimits in $\mathbf{\Theta}_n^{\mathsf{op}}$ and $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$, this comparison boils down to a homotopy equivalence between conically smooth diffeomorphisms of a stratified disk and the space of vari-framing on the disk. To establish this, we use that sections of the constructible tangent bundle can be integrated in families. This fully faithfulness of cellular realization then establishes the following pairing of concepts:

Category theory	**Stratified manifolds**
an (∞, n)-category $\mathcal{C}$	a vari-framed disk-stratified n-manifold M
a k-morphism	a connected k-dimensional stratum
composition of k-morphisms	merging k-dimensional strata by refinement
identity k-morphisms	creating a k-dimensional stratum
source & target maps	eliminating strata by closed morphisms

Section 4 is formal and short. As a result of §3, we can extend an (∞, n)-category $\mathcal{C}$ to a functor

$$\begin{tikzcd} \mathbf{\Theta}_n^{\mathsf{op}} \arrow[r, "\mathcal{C}"] \arrow[d] & \mathcal{S}\mathsf{paces} \\ \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \arrow[ur, dashed, "\mathcal{C}"'] & \end{tikzcd}$$

by right Kan extension along $\mathbf{\Theta}_n^{\mathsf{op}} \hookrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$, which we give the same notation. To define factorization homology $\int_M \mathcal{C}$ for a general vari-framed n-manifold M, we left Kan extend:

$$\begin{tikzcd} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \arrow[r, "\mathcal{C}"] \arrow[d] & \mathcal{S}\mathsf{paces} \\ \mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}} \arrow[ur, dashed, "\int \mathcal{C}"'] & \end{tikzcd} .$$

This completes the construction of the fully faithful functor $\mathsf{Cat}_{(\infty,n)} \longrightarrow \mathsf{Fun}(\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces})$.

Section 5 is an appendix concerning some checkable criteria in higher category theory. This appendix has four parts, each of which is entirely formal and contains essentially no new ideas. The first part establishes the notion of a *monomorphism* among ∞-categories, which we find to be a convenient way to articulate comparisons among ∞-categories. The second part records some easy facts about ∞-categories of cospans. The third part addresses when base change among ∞-categories is a left adjoint. This is an essential aspect of how we endow fiberwise structures on constructible families of stratified spaces. The fourth part simply explains how the univalence condition among Segal $\mathbf{\Theta}_n$-spaces is implied by lacking non-trivial higher idempotents.

Comparison with other works. Our notion of factorization homology is a direct generalization, from the $\mathcal{E}_n$-algebra case, of the labeled configuration spaces of Salvatore [Sa] and Segal [Se3]. These are both special cases of factorization homology, or topological chiral homology, after Lurie [Lu2]. An approach to manifold invariants via labeling fine stratifications of a manifold by a higher category was first well-demonstrated in 3-dimensions by Turaev–Viro [TV] as state-sums.

The essential notion underlying this work, of defining a homology theory by integrating over disk-stratifications, was earlier conceived by Morrison–Walker [MW]. This is likewise the essential notion for their blob homology. Our factorization homology is thus a spiritual cousin of their blob homology. However, there are significant differences in concept, execution, and result. We now survey these differences, with the dual purpose of discussing some features of our present framework that embody certain key developmental choices in our setup.

First, there is a difference in end result: we prove that an (∞, n)-category defines an input for factorization homology. To accomplish this takes the combined work of the present paper, of [AFR], and of [AFT], to fuse combinatorics and differential topology. It makes use of: the introduction of the

notion of conical smoothness of stratifications and a host of results about the differential topology thereof; the striation sheaf model of ∞-categories; the striation sheaf property of $\mathcal{B}\mathsf{un}$, showing existence of composition of morphisms via resolution of singularities; the homotopy equivalence between conically smooth diffeomorphisms of $\mathbb{D}^n$ and its space of vari-framings. Morrison & Walker have not yet shown that their blob homology can take as input an (∞,n)-category, with or without duals/adjoints. Instead, they conceive their own notion of an (∞,n)-category, with duals, and suggest that examples of interest will naturally fall within their framework.

A technical difference is that our definitions of homology are, in detail, quite different and not easily comparable. We define factorization homology as a colimit over an ∞-category $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{n/M}$ whose morphisms are compositions of four basic types: (1) refining strata, (2) creating strata, (3) isotoping strata, and (4) eliminating strata. Morrison & Walker define blob homology as a colimit over a poset $\mathfrak{D}(M)$ of stratifications of M, the morphisms in which are of type (1), namely refining strata. Also, factorization homology is naturally defined on the ∞-category of (∞,n)-categories – in particular, it is a homotopy invariant of an (∞,n)-category. In contrast, their work does not consider naturality in functors or homotopy-invariance in the higher category variable.

Variants of these four types of morphisms all occur in [MW], where they are called anti-refinements, pinched products, isotopies, and the boundary natural transformations. However, how to compose these maps, such as how to compose a refinement and pinched product, is not part of their schema. We address this problem of composition by verifying that the simplicial space determined by $\mathcal{B}\mathsf{un}$ satisfies the Segal condition; so composition is defined only up to coherent homotopy. Thereafter, the entities $\mathcal{D}\mathsf{isk}^{\tau}_n$ and $\mathcal{M}\mathsf{fd}^{\tau}_n$ are ∞-categories, thereby organizing the compositions of these types of morphisms. We do not think that homotopy coherent compositions of these classes of morphisms in these ∞-categories satisfy a universal property with respect to the individual classes, i.e., can be viewed as a condition. In particular, there is no four-term factorization system for these morphisms, and such a factorization system is still far weaker than an axiomatization. Consequently, we do not see how our model in terms of $\mathcal{B}\mathsf{un}$ and $\mathcal{M}\mathsf{fd}^{\tau}_n$ could be characterized by the axioms given at the end of §6.1 of [MW].

Regarding (3), another difficulty of comparison is the absence of common point-set refinements of two disk-stratifications. If one allows refining strata to be isotoped, then common refinements do exist. This leads to organizing morphisms in $\mathcal{B}\mathsf{un}$ as *spaces*, the paths in which account for isotopies. A lack of topology on mapping spaces would also obstruct any comparison with combinatorial models of (∞,n)-categories. In particular, we are able to define a fully faithful functor $\boldsymbol{\Theta}^{\mathsf{op}}_n \to \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_n$ exactly because the righthand side is topologized; without a topology on mapping spaces (e.g., allowing isotopies of stratifications as invertible morphisms) a discrete version of the righthand side—as is used in the blob setting—need not receive any functor from $\boldsymbol{\Theta}^{\mathsf{op}}_n$ or any other collection of combinatorial generators for (∞,n)-categories. One can work with a discrete category of refinements and take the Dwyer–Kan localization with respect to stratified isotopies, but this leads to the difficult problem of identifying this Dwyer–Kan localization. Further, it is unclear how one would match this approach to refinements with the other morphisms, such as creations morphisms: even if the ∞-subcategory $\mathcal{D}\mathsf{isk}^{\mathsf{ref}}_n$ of refinements among disk-stratified manifolds might be realized as a Dwyer–Kan localization of an ordinary category of refinements, it unclear how to even conjecturally extend this picture to the entire ∞-category $\mathcal{D}\mathsf{isk}_n$, the construction of which is inherently ∞-categorical and lacking strict compositions – see the verification of the Segal condition for $\mathcal{B}\mathsf{un}$ from [AFR].

A last difference stems from our introduction of the vari-framing. The rigid geometric structure of the vari-framing allows for factorization homology to take coefficients in (∞,n)-categories, rather than (∞,n)-categories with adjoints. If one used a more naive notion of a framing, such as a framing on the ambient manifold, then this would require the input (∞,n)-category to have adjoints. The existence of adjoints is an extremely restrictive condition on an (∞,n)-category, and becomes more restrictive as n increases, so this allows factorization homology to be defined with far more possible inputs.

Acknowledgements. JF and NR thank Alexander Beilinson for his inspiring mathematics and for his kind encouragement. DA thanks Richard Hepworth, discussions with whom informed our treatment of $\mathbf{\Theta}_n$. DA extends warm appreciation to Ana Brown for her persistent support.

1. Recollections of striation sheaves

In this section, we recall definitions and results from our antecedent works. The reference for §1.1 is [AFT], and the reference for the subsequent sections is [AFR]. This section is only an overview, so see those works for precise definitions and details; all the assertions below are substantiated in those works.

1.1. **Stratified spaces.** The work [AFT] presents a theory of smoothly stratified spaces founded on a key technical feature of conical smoothness. The conical smoothness condition prevents the existence of shearing maps. That is, a map of Whitney stratified spaces $X \to Y$ might be smooth on a closed stratum X_p as well as on its open complement $X \smallsetminus X_p$, yet still exhibit some degenerate shearing behavior as a path moves from $X \smallsetminus X_p$ into X_p. The essential local property that conical smoothness should have is that for a map between cones $F : \mathsf{C}(L) \to \mathsf{C}(M)$ such that $f^{-1}* = *$, there should exist a smooth isotopy so that f is equivalent to the cone on a map $L \to M$.

We give now give a rough definition of conical smoothness. The definition is by induction on dimension and depth of singularity type, starting with the case of maps between cones, so it will appear to be circular. See [AFT] for a detailed treatment.

First fixing notation, for $Z \to P$ a stratified space indexed by a poset P, the cone on Z is the space

$$\mathsf{C}(Z) := * \coprod_{Z\times\{0\}} Z \times [0,1) \qquad \longrightarrow \qquad P^{\triangleleft} := * \amalg_{P\times\{0\}} P \times \{0 < 1\}$$

stratified by the left-cone of the indexing poset P. The $*$-stratum consists the cone-point $* \subset \mathsf{C}(Z)$.

A *basic* is a stratified space of the form $\mathbb{R}^i \times \mathsf{C}(Z) = \left(\mathbb{R}^i \times \mathsf{C}(Z) \to \mathsf{C}(Z) \to P^{\triangleleft}\right)$ with $Z = (Z \to P)$ a compact stratified space. Each basic has a *cone-locus* as well as an *origin*:

$$\mathbb{R}^i \subset \mathbb{R}^i \times \mathsf{C}(Z) \qquad \text{and} \qquad 0 \in \mathbb{R}^i \subset \mathbb{R}^i \times \mathsf{C}(Z)\ .$$

A map between basics $f\colon \mathbb{R}^i \times \mathsf{C}(Y) \to \mathbb{R}^j \times \mathsf{C}(Z)$ is *conically smooth* if either the image of the source cone-locus does not intersect the target cone-locus and does so conically smoothly, or the map carries the cone-locus to the cone-locus and, for each $(p, s, z) \in \mathbb{R}^i \times \mathsf{C}(Y)$, and each $v \in \mathbb{R}^i$, the limit

$$\lim_{t\to 0} \frac{f(p + tv, ts, z)}{t} \quad \in \quad \mathbb{R}^j \times \mathsf{C}(Z) \tag{1}$$

exists and is again *conically smooth* in the arguments (p, s, z) and v. Since the topological dimension of Z is necessarily strictly less than that of X, this can be made well-defined in the induction.

We now give a similarly inductive definition of stratified spaces, in the conically smooth sense.

Definition 1.1 (Stratified spaces)**.** A singular manifold X is a paracompact Hausdorff topological space with a continuous map $X \to P$ to a poset, together with a maximal atlas by basics $\mathbb{R}^i \times \mathsf{C}(L)$ for which the transition maps are *conically smooth*. A *stratified space* X consists of a singular manifold $X \to P$ together with a surjective map of posets $P \to Q$. An object of the category Strat is a stratified spaces while a morphism is a *conically smooth* map $f\colon X \to Y$, by which it is meant a continuous map over a map of posets which is locally, with respect to the given atlases, *conically smooth*. Composition is given by composing continuous maps and composing maps of posets.

Here are a number of notable classes of morphisms in Strat.

Definition 1.2. Let $f\colon X \to Y$ be a conically smooth map of stratified spaces.

- **Embedding** (emb)**:** f is an *open embedding* if it is an isomorphism onto its image as well as open map of underlying topological spaces.

- **Refinement** (ref)**:** f is a *refinement* if it is a homeomorphism of underlying topological spaces, and, for each stratum $X_p \subset X$, the restriction $f_|\colon X_p \to Y$ is an isomorphism onto its image.
- **Open** (open)**:** f is *open* if it is an open embedding of underlying topological spaces and a refinement onto its image.
- **Fiber bundle:** f is a *fiber bundle* if, the collection of images $\phi(O) \subset Y$, indexed by pullback diagrams
$$\begin{tikzcd} F \times O \arrow[r] \arrow[d] & X \arrow[d] \\ O \arrow[r, "\phi"] & Y \end{tikzcd}$$
in which the horizontal maps are open embeddings, forms a basis for the topology of Y.
- **Constructible bundle** (cbl)**:** f is a *weakly constructible bundle* if, for each stratum $Y_q \subset Y$, the restriction $f_|\colon f^{-1}Y_q \to Y_q$ is a fiber bundle. The definition of a constructible bundle is inductive based on depth: in the base case of smooth manifolds, $f\colon X \to Y$ is a constructible bundle if it is a fiber bundle; in the inductive step of the definition, $f\colon X \to Y$ is a constructible bundle if it is a weakly constructible bundle and, additionally, if for each stratum $Y_q \subset Y$ the natural map
$$\mathsf{Link}_{f^{-1}Y_q}(X) \longrightarrow f^{-1}Y_q \underset{Y_q}{\times} \mathsf{Link}_{Y_q}(Y)$$
is a constructible bundle.
- **Proper constructible** ($\mathsf{p.cbl}$)**:** f belongs to the class ($\mathsf{p.cbl}$) if it is a constructible bundle and it is *proper*, i.e., if $f^{-1}C \subset X$ is compact for each compact subspace $C \subset Y$. f belongs to either of the classes ($\mathsf{p.cbl}, \mathsf{surj}$) or ($\mathsf{p.cbl}, \mathsf{inj}$) if it is proper constructible as well as either surjective or injective, respectively.

There are two natural cosimplicial stratified spaces.

Definition 1.3. The *extended* cosimplicial stratified space is the functor
$$\Delta^\bullet_e\colon \mathbf{\Delta} \longrightarrow \mathsf{Strat}~, \qquad [q] \mapsto \Delta^q_e := \bigl\{\{0,\dots,p\} \xrightarrow{t} \mathbb{R} \mid \sum_i t_i = 1\bigr\}$$
with the values on morphisms standard. The *standard* cosimplical stratified space is the functor
$$\Delta^\bullet\colon \mathbf{\Delta} \longrightarrow \mathsf{Strat}~, \qquad [p] \mapsto \Bigl(\Delta^p \ni t \mapsto \mathsf{Max}\{i \mid t_i \neq 0\} \in [p]\Bigr)$$
with the values on morphisms standard.

Remark 1.4. Note the isomorphism of stratified spaces $\Delta^q_e \cong \mathbb{R}^q$ with the smooth Euclidean space, as well as the isomorphism of stratified spaces $\Delta^p \cong \overline{\mathsf{C}}(\Delta^{p-1})$, where $\overline{\mathsf{C}}$ is the *closed* cone which is stratified by the left cone on the stratifying poset for Δ^{p-1}.

The extended cosimplicial stratified space accommodates a natural enrichment of Strat over Kan complexes.

Definition 1.5 ($\mathcal{S}\mathsf{trat}$)**.** The ∞-category $\mathcal{S}\mathsf{trat}$ is that associated to the Kan-enriched category for which an object is a stratified space and the Kan complex of morphisms from X to Y is the simplicial set
$$\mathcal{S}\mathsf{trat}(X,Y) ~:=~ \mathsf{Strat}_{/\Delta^\bullet_e}(X \times \Delta^\bullet_e, Y \times \Delta^\bullet_e)~;$$
composition is given by composition in Strat over $\Delta^\bullet_e$. There are a number of notable subsidiary ∞-categories
$$\mathcal{S}\mathsf{trat}^{\mathsf{ref}}~,~\mathcal{S}\mathsf{trat}^{\mathsf{emb}} ~\to~ \mathcal{S}\mathsf{trat}^{\mathsf{open}} ~\to~ \mathcal{S}\mathsf{trat} ~\leftarrow~ \mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}} ~\leftarrow~ \mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,surj}}~,~\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,inj}}$$
which are each associated to Kan enriched categories that have the same objects as Strat yet with Hom-Kan complexes consisting of those simplices of $\mathsf{Strat}_{/\Delta^\bullet_e}(-,-)$ which are fiberwise over $\Delta^\bullet_e$ of the indicated class.

(The regularity along strata ensured by conical smoothness can be used to verify that these Hom-simplicial sets are indeed Kan complexes.)

Manifest is a functor $\mathsf{Strat} \to \mathcal{S}\mathsf{trat}$ between ∞-categories. This functor has the property that, for each stratified space X, the morphism $X \times \mathbb{R} \xrightarrow{\mathsf{pr}} X$ in Strat is carried to an equivalence in $\mathcal{S}\mathsf{trat}$.

Theorem 1.6 ([AFR]). *The functor* $\mathsf{Strat} \to \mathcal{S}\mathsf{trat}$ *witnesses an equivalence between ∞-categories:*

$$\mathsf{Strat}[\mathbb{R}^1 \times -^{-1}] \xrightarrow{\simeq} \mathcal{S}\mathsf{trat}$$

from the localization on the collection of morphisms which are projections off of $\mathbb{R}$.

1.2. **Exit paths.** The enrichment $\mathcal{S}\mathsf{trat}$ of stratified spaces allows for a very natural presentation of the exit-path ∞-category of a stratified space, after Lurie [Lu2] and MacPherson. Following [AFR], we use the standard simplices to define the exit-path ∞-category $\mathsf{Exit}(X)$ as a complete Segal space. As a simplicial object, $\mathsf{Exit}(X)$ is the stratified version of the singular simplicial object $\mathsf{Sing}_\bullet(X)$.

Definition 1.7. The *exit-path ∞-category functor* is the restricted Yoneda functor

$$\mathsf{Exit}\colon \mathcal{S}\mathsf{trat} \longrightarrow \mathsf{PShv}(\mathbf{\Delta}) ~, \qquad X \mapsto \Big([p] \mapsto \mathcal{S}\mathsf{trat}(\Delta^p, X) = |\,\mathsf{Strat}(\Delta^p \times \Delta^\bullet_e, X)|\Big) ~.$$

The following is one of the main results of [AFR].

Theorem 1.8 ([AFR]). *The functor* Exit *factors fully faithfully through ∞-categories*

$$\mathsf{Exit}\colon \mathcal{S}\mathsf{trat} \longrightarrow \mathsf{Cat}_\infty ~, \tag{2}$$

incarnated here as complete Segal spaces. Also, the following diagrams in Strat *are colimit diagrams, and this functor (2) carries each of these diagrams to colimit diagrams among ∞-categories:*

- *Open hypercovering diagrams* $\mathcal{U}^{\triangleright} \to \mathsf{Strat} \to \mathcal{S}\mathsf{trat}$*;*
- *Blow-up diagrams*
$$\begin{tikzcd} \mathsf{Link}_{X_0}(X) \arrow[r] \arrow[d] & \mathsf{Unzip}_{X_0}(X) \arrow[d] \\ X_0 \arrow[r] & X; \end{tikzcd}$$
- *Iterated cone diagrams*
$$\begin{tikzcd} \mathsf{C}(\emptyset) \arrow[r, "\mathsf{C}(\emptyset\hookrightarrow L)"] \arrow[d] & \mathsf{C}(L) \arrow[d] \\ \mathsf{C}^2(\emptyset) \arrow[r, "\mathsf{C}^2(\emptyset\hookrightarrow L)"] & \mathsf{C}^2(L) ~; \end{tikzcd}$$
- *The univalence diagram*
$$\begin{tikzcd} \Delta^{\{0<2\}} \amalg \Delta^{\{1<3\}} \arrow[r] \arrow[d] & \Delta^{\{0<1<2<3\}} \arrow[d] \\ \Delta^{\{0=2\}} \amalg \Delta^{\{1=3\}} \arrow[r] & \ast. \end{tikzcd}$$

The main geometric ingredient supporting the proof of Theorem 1.8 is the next result. For $\mathcal{X}$ an ∞-category, the *∞-category of arrows (in $\mathcal{X}$)*

$$\mathsf{Ar}(\mathcal{X}) ~:=~ \mathsf{Fun}([1], \mathcal{X})$$

is that of functors from $[1]$.

Lemma 1.9 ([AFR]). *Let $X = (X \to P)$ be a stratified space. There is a natural identification of the maximal ∞-subgroupoid of* $\mathsf{Exit}(X)$

$$\mathsf{Exit}(X)^\sim ~\simeq~ \coprod_{p \in P} X_p$$

as the coproduct of the underlying spaces of the strata of X. For each pair of strata $p, q \in P$ there is an identification of the space of morphisms in $\mathsf{Exit}(X)$ *from the X_p cofactor to the X_q cofactor*

$$\mathsf{Ar}(\mathsf{Exit}(X))_{|X_p \times X_q} \ \simeq \ \mathsf{Link}_{X_p}(X_{|P^{p/}})_q$$

as the underlying space of the q-stratum of the link of X_p in the open stratified subspace $X_{|P^{p/}} \subset X$.

1.3. **Striation sheaves.**

Definition 1.10. The ∞-category $\mathcal{S}\mathsf{tri}$ of *striation sheaves* is the full ∞-subcategory of $\mathsf{PShv}(\mathsf{Strat})$ consisting of those presheaves $\mathcal{F}$ that are $\mathbb{R}^1$-invariant and that carry the opposites of each of the distinguished colimit diagrams of Theorem 1.8 to limit diagrams among spaces.

The standard cosimplicial stratified space $\mathbf{\Delta} \xrightarrow{\Delta^\bullet} \mathsf{Strat}$ yields the restriction functor $\mathcal{S}\mathsf{tri} \to \mathsf{PShv}(\mathbf{\Delta})$.

Theorem 1.11 ([AFR])**.** *Restriction along the cosimplicial stratified space $\Delta^\bullet$ determines an equivalence of ∞-categories*

$$\mathcal{S}\mathsf{tri} \ \simeq \ \mathsf{Cat}_\infty$$

between striation sheaves and ∞-category of ∞-categories, incarnated here as complete Segal spaces. This equivalence sends an ∞-category $\mathcal{C}$ to the presheaf on Strat *that takes the values*

$$K \mapsto \mathsf{Map}_{\mathsf{Cat}_\infty}(\mathsf{Exit}(K), \mathcal{C}) \ .$$

We use striation sheaves to make interesting ∞-categories by hand from smooth stratified geometry. The principal such object is the ∞-category $\mathcal{B}\mathsf{un}$, the construction of which we now indicate.

Definition 1.12 ($\mathcal{B}\mathsf{un}$ and $\mathcal{E}\mathsf{xit}$)**.** $\mathcal{B}\mathsf{un}$ is the presheaf on stratified spaces that classifies constructible bundles:

$$\mathcal{B}\mathsf{un}\colon K \mapsto |\{X \underset{\pi,\ \mathsf{cbl}}{\longrightarrow} K \times \Delta^\bullet_e\}| \ ,$$

the moduli space of constructible bundles over K. $\mathsf{c}\mathcal{B}\mathsf{un}$ is the subpresheaf on stratified spaces that classifies *proper* constructible bundles:

$$\mathsf{c}\mathcal{B}\mathsf{un}\colon K \mapsto |\{X \underset{\pi,\ \mathsf{p.cbl}}{\longrightarrow} K \times \Delta^\bullet_e\}| \ ,$$

the moduli space of proper constructible bundles.
$\mathcal{E}\mathsf{xit}$ is the presheaf on stratified spaces that classifies constructible bundles equipped with a section:

$$\mathcal{E}\mathsf{xit}\colon K \mapsto \big|\big\{X \underset{\pi,\ \mathsf{cbl}}{\overset{\sigma}{\leftrightarrows}} K \times \Delta^\bullet_e \mid \sigma\pi = 1\big\}\big| \ .$$

The cumulative result of the work of [AFR], and of all the regularity around substrata ensured by conical smoothness, is the following.

Theorem 1.13 ([AFR])**.** *The presheaves* $\mathcal{B}\mathsf{un}$ *and* $\mathsf{c}\mathcal{B}\mathsf{un}$ *and* $\mathcal{E}\mathsf{xit}$ *are striation sheaves and, consequently, form ∞-categories via the equivalence* $\mathcal{S}\mathsf{tri} \underset{\text{Thm 1.11}}{\simeq} \mathsf{Cat}_\infty$. *Furthermore, the natural functor*

$$\mathsf{c}\mathcal{B}\mathsf{un} \longrightarrow \mathcal{B}\mathsf{un}$$

is a monomorphism of ∞-categories.

Remark 1.14. The ∞-subcategory $\mathsf{c}\mathcal{B}\mathsf{un} \subset \mathcal{B}\mathsf{un}$ consists only of *compact* stratified spaces, yet is *not* full. For instance, there is no morphism from $\emptyset$ to S^1 in $\mathsf{c}\mathcal{B}\mathsf{un}$, while there is a unique such morphism in $\mathcal{B}\mathsf{un}$.

Forgetting sections defines a functor $\mathcal{E}\mathsf{xit} \to \mathcal{B}\mathsf{un}$. The next result in particular identifies the fibers of this functor.

Proposition 1.15 ([AFR]). *For each functor* $\mathsf{Exit}(K) \xrightarrow{(X \xrightarrow{\pi} K)} \mathcal{B}\mathsf{un}$ *classifying the indicated constructible bundle, the diagram among* ∞*-categories*

$$\begin{tikzcd}
\mathsf{Exit}(X) \arrow[r, "(X \underset{K}{\times} X \underset{\mathsf{pr}}{\overset{\mathsf{diag}}{\leftrightarrows}} X)"] \arrow[d, "\mathsf{Exit}(\pi)"'] & \mathcal{E}\mathsf{xit} \arrow[d] \\
\mathsf{Exit}(K) \arrow[r, "(X \xrightarrow{\pi} K)"] & \mathcal{B}\mathsf{un}
\end{tikzcd}$$

is a pullback diagram.

The reader might notice that the above description of $\mathcal{B}\mathsf{un}$, as well as each variant, is derived from more primitive data. We explain this, for we make use of such a maneuver in §3.1 of this article.

Definition 1.16 (§6 of [AFR]). The ordinary category Bun is that for which an object is a constructible bundle $X \to K$ and a morphism from $(X \to K)$ to $(X' \to K')$ is a pullback diagram among stratified spaces

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d] \\
K \arrow[r] & K'.
\end{tikzcd}$$

Composition is given by concatenating such diagrams horizontally and composing horizontal arrows.

In §6 of [AFR] we prove that the projection

$$\mathsf{Bun} \longrightarrow \mathsf{Strat}~, \qquad (X \to K) \mapsto K$$

is a right fibration, which we can straighten to a functor

$$\mathsf{Strat}^{\mathsf{op}} \longrightarrow \mathsf{Gpd}$$

to the ordinary category of groupoids. The presheaf $\mathcal{B}\mathsf{un}$ can be obtained as the functor

$$\mathcal{B}\mathsf{un}\colon \mathsf{Strat}^{\mathsf{op}} \longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Gpd}) \xrightarrow{|-|} \mathcal{S}\mathsf{paces}~, \qquad X \mapsto |\,\mathsf{Bun}(X \times \Delta^{\bullet}_e)|$$

where here the arrow labeled as $|-|$ is a standard nerve functor. More generally, for $\mathfrak{F} \in \mathsf{Fun}(\mathsf{Strat}^{\mathsf{op}}, \mathsf{Gpd})$ a groupoid-valued functor, the expression $\mathcal{F}\colon X \mapsto |\mathfrak{F}(X \times \Delta^{\bullet}_e)|$ defines a space-valued presheaf. This association $\mathfrak{F} \mapsto \mathcal{F}$ is the *topologizing diagram* functor, and is developed in §2 of [AFR].

1.4. **Classes of morphisms.** Here we name several classes of morphisms in $\mathcal{B}\mathsf{un}$. To identify these morphisms, it is convenient to use that morphisms in the ∞-category $\mathcal{B}\mathsf{un}$ can be constructed as mapping cylinders of stratified maps in two different ways, as cylinders of open maps or as reversed cylinders of proper constructible maps.

Theorem 1.17. *There are functors*

$$\mathcal{S}\mathsf{trat}^{\mathsf{open}} \longrightarrow \mathcal{B}\mathsf{un} \longleftarrow (\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}})^{\mathsf{op}}$$

each of which is a monomorphism.

This allows for the following definition.

Definition 1.18.
- The ∞-subcategory $\mathcal{B}\mathsf{un}^{\mathsf{cls}} \subset \mathcal{B}\mathsf{un}$ of *closed* morphisms is the image of $(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,inj}})^{\mathsf{op}}$.
- The ∞-subcategory $\mathcal{B}\mathsf{un}^{\mathsf{cr}} \subset \mathcal{B}\mathsf{un}$ of *creation* morphisms is the image of $(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,surj}})^{\mathsf{op}}$.
- The ∞-subcategory $\mathcal{B}\mathsf{un}^{\mathsf{ref}} \subset \mathcal{B}\mathsf{un}$ of *refinement* morphisms is the image of $\mathcal{S}\mathsf{trat}^{\mathsf{ref}}$.
- The ∞-subcategory $\mathcal{B}\mathsf{un}^{\mathsf{emb}} \subset \mathcal{B}\mathsf{un}$ of *open embedding* morphisms is the image of $\mathcal{S}\mathsf{trat}^{\mathsf{emb}}$.

- The ∞-subcategory $\mathcal{B}\mathsf{un}^{\mathsf{act}} \subset \mathcal{B}\mathsf{un}$ of *active* morphisms is the smallest containing the creation and refinement and embedding morphisms.

Remark 1.19. The intersection of the ∞-subcategories $\mathsf{c}\mathcal{B}\mathsf{un} \cap \mathcal{B}\mathsf{un}^{\mathsf{emb}} \simeq \mathsf{c}\mathcal{B}\mathsf{un}^{\sim}$ is the maximal ∞-subgroupoid of $\mathcal{B}\mathsf{un}$ consisting of the compact stratified spaces.

We note these classes of morphisms have an equivalent definition in terms properties of *links*. That is, consider a morphism $X_0 \to X_1$ in $\mathcal{B}\mathsf{un}$ represented by a constructible bundle $X \to \Delta^1$. The morphism is closed (respectively, a creation) if the natural map $\mathsf{Link}_{X_0}(X) \to X_0$ is an embedding (respectively, is surjective) and the open conically smooth map

$$\mathsf{Link}_{X_0}(X) \times [0,1) \cong \mathsf{Unzip}_{X_0}(X)_{|\,\mathsf{Link}_{\{0\}}(\Delta^1)\times[0,1)}$$

can be chosen to be an isomorphism. This is the geometry behind the following result.

Lemma 1.20 ([AFR])**.** *The pair of ∞-subcategories $(\mathcal{B}\mathsf{un}^{\mathsf{cls}}, \mathcal{B}\mathsf{un}^{\mathsf{act}})$ is a factorization system on $\mathcal{B}\mathsf{un}$.*

Recall the unstraightening construction from §3.2 of [Lu2] which constructs a monomorphism

$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Cat}_\infty) \longrightarrow \mathsf{Cat}_{\infty\,/\mathcal{D}}~.$$

The essential image consists of *Cartesian fibrations* over $\mathcal{D}$.

Lemma 1.21. *Each solid diagram among ∞-categories*

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{E}\mathsf{xit} \arrow[d] \\ {[1]} \arrow[r, "o"] \arrow[ur, dashed] & \mathcal{B}\mathsf{un} \end{tikzcd}$$

in which o classifies either a refinement morphism or an embedding morphism of $\mathcal{B}\mathsf{un}$ admits a filler that classifies a coCartesian morphism over $\mathcal{B}\mathsf{un}$. Each solid diagram among ∞-categories

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \mathcal{E}\mathsf{xit} \arrow[d] \\ {[1]} \arrow[r, "c"] \arrow[ur, dashed] & \mathcal{B}\mathsf{un} \end{tikzcd}$$

in which c classifies either a closed morphism or a creation morphism of $\mathcal{B}\mathsf{un}$ admits a filler that classifies a Cartesian morphism over $\mathcal{B}\mathsf{un}$.

Proof. The first statement follows because $\mathcal{E}\mathsf{xit}_{|\,\mathcal{B}\mathsf{un}^{\mathsf{ref,emb}}} \to \mathcal{B}\mathsf{un}^{\mathsf{ref,emb}} \simeq \mathcal{S}\mathsf{trat}^{\mathsf{open}}$ is the unstraightening of the functor

$$\mathsf{Exit}\colon \mathcal{S}\mathsf{trat}^{\mathsf{open}} \longrightarrow \mathsf{Cat}_\infty~.$$

The second statement follows because $\mathcal{E}\mathsf{xit}_{|\,\mathcal{B}\mathsf{un}^{\mathsf{cls,cr}}} \to \mathcal{B}\mathsf{un}^{\mathsf{cls,cr}} \simeq (\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}})^{\mathsf{op}}$ is the unstraightening of the functor

$$\mathsf{Exit}\colon ((\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}})^{\mathsf{op}})^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty~.$$

□

We mirror Definition 1.18 with classes of morphisms in $\mathcal{E}\mathsf{xit}$. We do this in a way that reflects the natural handedness of the various restrictions $\mathcal{E}\mathsf{xit}_{|\,\mathcal{B}\mathsf{un}^{\psi}} \to \mathcal{B}\mathsf{un}^{\psi}$.

Definition 1.22.

- The ∞-subcategory $\mathcal{E}\mathsf{xit}^{\mathsf{cls}} \subset \mathcal{E}\mathsf{xit}$ of *closed* morphisms consists of those morphisms in $\mathcal{E}\mathsf{xit}$ that are Cartesian over $\mathcal{B}\mathsf{un}$ and whose image in $\mathcal{B}\mathsf{un}$ is a closed morphism.
- The ∞-subcategory $\mathcal{E}\mathsf{xit}^{\mathsf{cr}} \subset \mathcal{E}\mathsf{xit}$ of *creation* morphisms consists of those morphisms in $\mathcal{E}\mathsf{xit}$ that are Cartesian over $\mathcal{B}\mathsf{un}$ and whose image in $\mathcal{B}\mathsf{un}$ is a creation morphism.
- The ∞-subcategory $\mathcal{E}\mathsf{xit}^{\mathsf{ref}} \subset \mathcal{E}\mathsf{xit}$ of *refinement* morphisms consists of those morphisms in $\mathcal{E}\mathsf{xit}$ that are coCartesian over $\mathcal{B}\mathsf{un}$ and whose image in $\mathcal{B}\mathsf{un}$ is a refinement morphism.

- The ∞-subcategory $\mathcal{E}\mathsf{xit}^{\mathsf{emb}} \subset \mathcal{E}\mathsf{xit}$ of *embedding* morphisms consists of those morphisms in $\mathcal{E}\mathsf{xit}$ that are coCartesian over $\mathcal{B}\mathsf{un}$ and whose image in $\mathcal{B}\mathsf{un}$ is an embedding morphism.

2. Tangential structures

The ∞-category of constructible bundles is vast. We are primarily interested in the ∞-subcategory of $\mathcal{B}\mathsf{un}$ classifying constructible bundles whose fibers are stratified n-manifolds, as well as variations which account for infinitesimal structures thereon. We recognize these entities as $\mathcal{B}\mathsf{un}^{\tau}$ for appropriately chosen tangential structures τ. Forgetting structure defines a functor $\mathcal{B}\mathsf{un}^{\tau} \to \mathcal{B}\mathsf{un}$; this functor will have partial fibration properties, mirroring the definition of ∞-operads as functors $\mathcal{O} \to \mathsf{Fin}_*$ as developed in [Lu2] (Definition 2.1.1.10). In this section, we give a general framework for such structures; manipulations among them will be key for the main results of this article. We begin by generalizing the standard tangent bundle of smooth manifolds.

2.1. **Constructible tangent bundle.**

Here we define the *constructible* tangent bundle of a stratified space: $TX \to X$. For a treatment in the context of Whitney stratified spaces, see §2.1 of [Pf]. Our treatment is tailored to accommodate a number of functorialities which are akin to parallel transport through strata as well as variation in constructible families. The main outcome of this section is the Definition 2.13 of the *constructible tangent bundle*

$$\mathsf{T}\colon \mathcal{E}\mathsf{xit} \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}} ~,$$

which will be defined and explained.

2.1.1. **Synopsis.**

In essence, we define the constructible tangent bundle of a stratified space X by naming a sheaf of vector spaces on X, the stalk-dimensions of which are finite and lower semi-continuous. In terms of parallel transport systems, such data is classified by a functor

$$\mathsf{T}_X : \mathsf{Exit}(X) \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$$

from the exit-path ∞-category of X to an ∞-category of vector spaces and injections among them. In the case that $X = M$ is an ordinary smooth manifold, this is the sheaf of vector fields on M, which we might suggestively think of as infinitesimal automorphisms of X, and the functor $\mathsf{Exit}(M) \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ is the classifying map of its tangent bundle $M \to \coprod_{i\geq 0} \mathsf{BO}(i)$.

We thus seek an assignment of each stratified space X to a functor $\mathsf{Exit}(X) \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$. We do this in such as way as to reveal an assortment of functorialities of this assignment among such X. To learn what functorialities we might expect we can consider the case of ordinary smooth manifolds. Namely, a smooth map $f\colon M \to N$ between ordinary smooth manifolds determines a map of vector bundles $Df\colon TM \to f^*TN$ on M. This map is an isomorphism whenever f is an open embedding, but otherwise it typically will not be an isomorphism. Therefore we expect functoriality for the constructible tangent bundle with respect to open embeddings among stratified spaces. Thinking of vector fields as infinitesimal automorphisms of a manifold, we also expect that the constructible tangent bundle of a stratified space restricts along strata: for $g\colon X_0 \hookrightarrow X$ the inclusion of a stratum, we expect an isomorphism of vector bundles $Dg\colon TX_0 \cong g^*TX$ on X_0. This latter expected functoriality does not appear in the case of ordinary smooth manifolds. (Note the distinction between this indication of *tangent* and that supplied by algebraic geometry whose sections cannot be integrated when X is not smooth, even in characteristic zero.)

In our Definition 2.13, the constructible tangent bundle will be a functor between ∞-categories

$$\mathsf{T}\colon \mathcal{E}\mathsf{xit} \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}} ~.$$

For each stratified space X, there is a natural functor $\mathsf{Exit}(X) \to \mathcal{E}\mathsf{xit}$, and this functor T will then specialize to the expected functor $\mathsf{T}_X\colon \mathsf{Exit}(X) \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$. Furthermore, the functor T specializes along the monomorphisms $(\mathcal{S}\mathsf{trat}^{*/})^{\mathsf{emb}} \hookrightarrow \mathcal{E}\mathsf{xit} \hookleftarrow ((\mathcal{S}\mathsf{trat}^{*/})^{\mathsf{p.cls.inj}})^{\mathsf{op}}$ as the expected functorialities. In this way, our definition of the constructible tangent bundle as the functor T accommodates our intuition.

It is awkward to define this constructible tangent bundle directly. Simply requiring it to possess the expected functorialities and to restrict to the familiar notion for smooth manifolds completely characterizes it. So we articulate a sense in which structures on stratified spaces that are suitably local are completely characterized by their values on Euclidean spaces.

Remark 2.1. Our Definition 2.13 of the constructible tangent bundle as a functor between ∞-categories $\mathcal{E}\mathsf{xit} \xrightarrow{\mathsf{T}} \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ manifestly carries isotopies among stratified spaces, such as manifolds, to equivalences of vector bundle data. This phrasing greatly consolidates tangential data, but it also loses a considerable amount of infinitesimal geometric information. For instance, for M a smooth manifold, as a $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$-valued sheaf, T_M is simply a kind of local system of vector spaces on M, which one can regard as a continuous parallel transport system of vector spaces on M. Of course, such parallel transport on the tangent bundle of M is only defined upon a choice of connection, which has rich infinitesimal geometry. Because the space of connections is contractible (for it is convex), the phrasing in terms of the ∞-category $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ collapses this choice.

2.1.2. **Vector spaces.** Here we define a constructible sheaf $\mathcal{V}\mathsf{ect}$ on stratified spaces that classifies stratum-wise vector bundles, in which the dimensions of the fibers may vary across strata. We first define conically smooth vector bundles, in which the dimensions of the fibers are locally constant across strata. (All vector spaces are understood to be finite dimensional real vector spaces.)

Definition 2.2. A *conically smooth vector bundle* $V \to K$ is a conically smooth map of stratified spaces $V \to K$ together with

- a conically smooth section $K \xrightarrow{0} V$;
- a conically smooth map $V \underset{K}{\times} V \xrightarrow{+} V$ over K;
- a conically smooth map $\mathbb{R} \times V \to V$ over K.

These data satisfy the following locality condition.

- There is an open cover $\mathcal{U}$ of K for which, for each $U \in \mathcal{U}$, there is a vector space $V_U = (0 \in V_U, +, \cdot)$ and an isomorphism from the restricted data over U:
$$(U \xrightarrow{0_{|U}} V_{|U}\ ,\ +_{|U}\ ,\ \cdot_{|U})\ \cong\ (U \xrightarrow{\mathsf{id}_U \times \{0\}} U \times V_U\ ,\ \mathsf{id}_U \times +\ ,\ \mathsf{id}_U \times \cdot)\ .$$

A *map of vector bundles* from $(V \to K)$ to $(W \to L)$ is a commutative square among stratified spaces

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d] & W \arrow[d] \\ K \arrow[r, "f"] & L \end{tikzcd}$$

for which, for each $x \in K$, the map of fibers $F_{|x} \colon V_{|x} \to W_{|f(x)}$ is a linear map between vector spaces. Such a map of vector bundles has *locally constant rank* if $\mathsf{Ker}(F) \subset V$ is a sub-vector bundle over K.

Note that conically smooth vector bundles, and maps among them, form a category in which composition is given by concatenating such squares horizontally and composing horizontal arrows.

Remark 2.3. We point out that each conically smooth vector bundle $V \to K$ forgets to a vector bundle on the underlying topological space of K. In fact, just as smooth vector bundles suitably approximate vector bundles over smooth manifolds, conically smooth vector bundles suitable approximate vector bundles over stratified spaces.

In the next definition, we make use of the standard nerve functor $\mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Gpd}) \xrightarrow{|-|} \mathcal{S}\mathsf{paces}$ from simplicial groupoids to the ∞-category of spaces.

Definition 2.4 ($\mathcal{V}\mathsf{ect}$)**.** The simplicial space $\mathcal{V}\mathsf{ect} \colon \mathbf{\Delta}^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ is given by
$$[p]\ \mapsto\ \left|\left\{V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \dots \xrightarrow{f_p} V_p \text{ over } \Delta^{\bullet}_e\right\}\right|\ ,$$

the space of composable sequences of morphisms of vector bundles over the cosimplicial smooth manifold $\Delta^\bullet_e$ for which, for each $0 \leq i \leq j \leq p$, the composition $f_j \circ \cdots \circ f_i$ has locally constant rank. The simplicial subspaces

$$\mathcal{V}\mathsf{ect}^\sim \ \subset\ \mathcal{V}\mathsf{ect}^{\mathsf{inj}}\ ,\ \mathcal{V}\mathsf{ect}^{\mathsf{surj}} \ \subset\ \mathcal{V}\mathsf{ect}$$

consist, for each $p \geq 0$, of those p-simplices $V_0 \xrightarrow{f_1} \dots \xrightarrow{f_p} V_p$ for which each f_i is a fiberwise isomorphism, a fiberwise injection, and a fiberwise surjection, respectively.

Observation 2.5. The simplicial spaces $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ and $\mathcal{V}\mathsf{ect}^{\mathsf{surj}}$ are complete Segal spaces, and therefore present ∞-categories. The simplicial space $\mathcal{V}\mathsf{ect}^\sim$ is a constant simplicial space, and therefore presents an ∞-groupoid. This ∞-groupoid $\mathcal{V}\mathsf{ect}^\sim$ is identified as the maximal ∞-subgroupoid of both $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ and $\mathcal{V}\mathsf{ect}^{\mathsf{surj}}$. The simplicial space $\mathcal{V}\mathsf{ect}$ does *not* satisfy the Segal condition; therefore we do not regard it as an ∞-category.

Observation 2.6. There is a natural identification of ∞-groupoids

$$\coprod_{n\geq 0} \mathsf{BO}(n) \ \simeq\ \mathcal{V}\mathsf{ect}^\sim\ .$$

For i and j dimensions, there is an identification of the space of 1-simplices from $\mathbb{R}^i$ to $\mathbb{R}^j$

$$\mathcal{V}\mathsf{ect}(\mathbb{R}^i,\mathbb{R}^j) \ \simeq\ \coprod_{0\leq d\leq i,j} \big(\mathsf{O}(i)/\mathsf{O}(i-d)\big) \underset{\mathsf{O}(d)}{\times} \big(\mathsf{O}(j)/\mathsf{O}(j-d)\big)\ .$$

In particular, there are natural identifications of the spaces of morphisms

$$\mathcal{V}\mathsf{ect}^{\mathsf{inj}}(\mathbb{R}^i,\mathbb{R}^j) \ \simeq\ \mathsf{O}(j)/\mathsf{O}(j-i) \qquad \text{and} \qquad \mathcal{V}\mathsf{ect}^{\mathsf{surj}}(\mathbb{R}^i,\mathbb{R}^j) \ \simeq\ \mathsf{O}(i)/\mathsf{O}(i-j)\ .$$

Also, the zero vector space is initial in the ∞-category $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$, final in the ∞-category $\mathcal{V}\mathsf{ect}^{\mathsf{surj}}$.

Remark 2.7. Taking fiberwise linear duals implements an equivalence $(-)^\vee\colon \mathcal{V}\mathsf{ect}^{\mathsf{op}} \simeq \mathcal{V}\mathsf{ect}$. This equivalence restricts to $\mathcal{V}\mathsf{ect}^\sim$ as the identity, and it restricts as an equivalence $(\mathcal{V}\mathsf{ect}^{\mathsf{inj}})^{\mathsf{op}} \simeq \mathcal{V}\mathsf{ect}^{\mathsf{surj}}$.

Remark 2.8. We find it convenient to sometimes work with the simplicial space $\mathcal{V}\mathsf{ect}$ even though it does not present an ∞-category. Indeed, as in §1 of [Lu1], the notion of a limit diagram, and of a zero-object, in a simplicial space has meaning once the notion of a final object in a simplicial space has meaning.

> Let $\mathcal{V}$ be a simplicial space. Let $[0] \xrightarrow{V} \mathcal{V}$ be an object. The simplicial space $\mathcal{V}_{/V}$ is that for which $\mathsf{Map}([p],\mathcal{V}_{/V}) :\simeq \mathsf{Map}\big([p+1],\{p+1\}),(\mathcal{V},V)\big)$ for each $p\geq 0$. The object $V\in\mathcal{V}$ is *final* if the manifest projection $\mathcal{V}_{/V}\to\mathcal{V}$ is an equivalence of simplicial spaces.

Observation 2.9. The zero vector space is a zero-object of the simplicial space $\mathcal{V}\mathsf{ect}$. Furthermore, each map $[1]\to\mathcal{V}\mathsf{ect}$, which classifies a linear map $V\xrightarrow{f} W$ between vector spaces, canonically extends to a limit diagram $[1]\times[1]\to\mathcal{V}\mathsf{ect}$:

$$\begin{tikzcd} \mathsf{Ker}(f) \arrow[r, "i"] \arrow[d] & V \arrow[d, "f"] \\ 0 \arrow[r] & W \end{tikzcd}$$

in which i is an injection. Using Remark 2.7, should f be surjective then the above square diagram in $\mathcal{V}\mathsf{ect}$ is also a pushout, and there is a natural equivalence $\mathsf{Ker}(i)^\vee \simeq W$.

Fiberwise direct sum of vector spaces defines a symmetric monoidal structure on $\mathcal{V}\mathsf{ect}$,

$$\mathcal{V}\mathsf{ect}\ \in\ \mathsf{Alg}_{\mathsf{Com}}\big(\mathsf{PShv}(\boldsymbol{\Delta})^\times\big)\ ,$$

as we now explain. First note that the simplicial space $\mathcal{V}\mathsf{ect}$ is defined as the nerve of a simplicial object in simplicial groupoids. Fiberwise direct sum of vector bundles lifts this simplicial object in simplicial groupoids to a simplicial object in symmetric monoidal simplicial groupoids:

$$\bigoplus\colon \big((V\to\Delta^\bullet_e),(W\to\Delta^\bullet_e)\big) \ \mapsto\ \big(V\underset{\Delta^\bullet_e}{\times} W\to\Delta^\bullet_e\big)\ .$$

The existence of the proposed symmetric monoidal structure follows because geometric realization commutes with products.

Observation 2.10. The symmetric monoidal structure on $\mathcal{V}\mathsf{ect}$ restricts as symmetric monoidal structures on each of the ∞-categories $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$, $\mathcal{V}\mathsf{ect}^{\mathsf{surj}}$, and $\mathcal{V}\mathsf{ect}^{\sim}$.

2.1.3. **The constructible tangent bundle.** In §2 of [AFR] we considered the *topologizing diagram functor*

(3) $$\mathsf{Trans}\xrightarrow{\mathfrak{F}\mapsto|\mathfrak{F}(-\times\Delta^\bullet_e)|}\mathcal{S}\mathsf{tri}\underset{\text{[AFR]}}{\simeq}\mathsf{Cat}_\infty$$

from transversality sheaves, which are certain right fibrations $\mathfrak{F}\to\mathsf{Strat}$ between ordinary categories, to striation sheaves.

Observation 2.11. Because geometric realization commutes with finite products, the topologizing diagram functor (3) preserves finite products.

Corollary 2.12. *The topologizing diagram functor (3) carries commutative algebras in* Trans *to symmetric monoidal ∞-categories. In particular, the following ∞-categories have natural symmetric monoidal structures:*

$$\mathcal{B}\mathsf{un}\quad\text{with}\quad\big((X\to K),(Y\to K)\big)\mapsto\big(X\underset{K}{\times}Y\to K\big)\ ;$$

$$\mathsf{c}\mathcal{B}\mathsf{un}\quad\text{with}\quad\big((X\to K),(Y\to K)\big)\mapsto\big(X\underset{K}{\times}Y\to K\big)\ ;$$

$$\mathcal{E}\mathsf{xit}\quad\text{with}\quad\big((X\overset{\sigma}{\leftrightarrows}K),(Y\overset{\tau}{\leftrightarrows}K)\big)\mapsto\big(X\underset{K}{\times}Y\overset{\sigma\times\tau}{\leftrightarrows}K\big)\ .$$

The poset $\mathbb{Z}_{\geq0}$ of non-negative integers carries a natural symmetric monoidal structure given by addition. Notice the functor

$$\mathsf{dim}\colon\ \mathcal{E}\mathsf{xit}\longrightarrow\mathbb{Z}_{\geq0}\ ,\qquad\Big(\mathsf{Exit}(K)\xrightarrow[\pi]{(X\overset{\sigma}{\leftrightarrows}K)}\mathcal{E}\mathsf{xit}\Big)\ \mapsto\ \Big(\mathsf{Exit}(K)\xrightarrow{k\mapsto\mathsf{dim}_{\sigma(k)}((\pi^{-1}k)_p)}\mathbb{Z}_{\geq0}\Big)$$

given by assigning to each such indicated K-point of $\mathcal{E}\mathsf{xit}$ the functor $\mathsf{Exit}(K)\to\mathbb{Z}_{\geq0}$ whose value on $k\in K$ is the local dimension of the stratum of the fiber in which $\sigma(k)$ lies. Explicitly, in the case that $K=*$ is a point, this functor assigns to each pointed stratified space $(x\in X)$ the local dimension $\mathsf{dim}_x(X_p)$ of the stratum $X_p\subset X$ in which x lies. This functor is naturally a symmetric monoidal functor.

Notice the functor

$$\mathsf{dim}\colon\ \mathcal{V}\mathsf{ect}^{\mathsf{inj}}\longrightarrow\mathbb{Z}_{\geq0}$$

that carries each vector space to its dimension; this functor is naturally a symmetric monoidal functor. Notice the functor between ∞-categories

(4) $$\mathcal{V}\mathsf{ect}^{\sim}\longrightarrow\mathcal{E}\mathsf{xit}\ ,\qquad|(V\to\Delta^p\times\Delta^\bullet_e)|\mapsto|(V\overset{\mathsf{zero}}{\leftrightarrows}\Delta^p\times\Delta^\bullet_e)|\ ,$$

which is expressed here as a map of simplicial spaces, Because the underlying stratified space of the direct sum of two vector spaces agrees with the product of their underlying stratified spaces, this last functor is naturally a symmetric monoidal functor. Because vector space dimension equals topological dimension, this symmetric monoidal functor naturally lies over the symmetric monoidal ∞-category $\mathbb{Z}_{\geq0}$.

We are now prepared to present our definition of the constructible tangent bundle. The definition relies on a characterization, which is Proposition 2.14 below, which we prove in §2.3.3.

Definition 2.13. The *constructible tangent bundle* is the unique symmetric monoidal functor under $\mathcal{V}\mathsf{ect}^{\sim}$

$$\mathsf{T}\colon \mathcal{E}\mathsf{xit} \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$$

that carries both closed morphisms and embedding morphisms to equivalences.

Proposition 2.14. *The constructible tangent bundle exists and is uniquely characterized by the conditions of Definition 2.13.*

There is a functor

$$\epsilon^{\bullet}\colon \mathbb{Z}_{\geq 0} \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}} ~, \qquad (i\leq j)\mapsto (\mathbb{R}^i \xrightarrow{\mathsf{inc}} \mathbb{R}^j)$$

taking values in Euclidean vector spaces and standard inclusions among them as the first coordinates.

Definition 2.15. The *dimension bundle* is the composite functor

$$\epsilon^{\mathsf{dim}}\colon~ \mathcal{E}\mathsf{xit} \xrightarrow{\mathsf{T}} \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \xrightarrow{\mathsf{dim}} \mathbb{Z}_{\geq 0} \xrightarrow{\epsilon^{\bullet}} \mathcal{V}\mathsf{ect}^{\mathsf{inj}}~.$$

Notation 2.16. Let $\eta\colon~\mathcal{E}\mathsf{xit}\to\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ be a functor. For each constructible bundle $X\to K$, the functor

$$\eta^{\mathsf{v}}_X : \mathsf{Exit}(X) \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$$

is the restriction of η along the functor $\mathsf{Exit}(X)\to\mathcal{E}\mathsf{xit}$ classifying $(X \underset{K}{\times} X \underset{\mathsf{pr}}{\overset{\mathsf{diag}}{\leftrightarrows}} X)$. (See Proposition 1.15.) For each stratified space X, the functor

$$\eta_X\colon~\mathsf{Exit}(X)\longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$$

is the instance of the functor η^{v}_X applied to the constructible bundle $X\to *$.

2.1.4. **Elucidating T.** We postpone the construction of the constructible tangent bundle T to §2.3.3. Here we provide a simple description of T value-wise, without taking care to manage its coherent functoriality. Let $X=(X\to P)$ be a stratified space. Recall that the underlying ∞-groupoid of $\mathsf{Exit}(X)$ is the coproduct of spaces $\coprod\limits_{p\in P} X_p$; each summand is the underlying space of a smooth manifold. Recall that the space of morphisms of $\mathsf{Exit}(X)$ from the X_p component to the X_q component is the space L_{pq}, which is the q-stratum of the link of $X_p\subset X$; it is the underlying space of a smooth manifold. This smooth manifold is equipped as a smooth proper fiber bundle $X_p \xleftarrow{\pi_{pq}} L_{pq}$ as well as a smooth open embedding $L_{pq}\times(0,1)\xrightarrow{\gamma_{pq}} X_q$ – these data are determined by the stratified space X, up to contractible choices.

The functor $\mathsf{T}_X\colon~\mathsf{Exit}(X)\to\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ can be described on objects and morphisms as follows.

- **Objects:** It restricts to the component X_p of the underlying ∞-groupoid as the local system

$$TX_p\colon X_p\longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$$

classifying the tangent bundle of the smooth manifold X_p.
- **Morphisms:** It restricts to the component L_{pq} of the mapping space as the diagram of local systems

$$\begin{tikzcd}
L_{pq} \arrow[r, "D\gamma_{pq}\circ\mathsf{Z}\circ D\pi^{\vee}_{pq}"] \arrow[d, "(\pi_{pq}{,}\gamma_{pq|})"'] & \mathsf{Ar}(\mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \arrow[d, "(\mathsf{ev}_s{,}\mathsf{ev}_t)"] \\
X_p\times X_q \arrow[r, "TX_p\times TX_q"] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}}\times\mathcal{V}\mathsf{ect}^{\mathsf{inj}}
\end{tikzcd}$$

where $L_{pq}\to\mathsf{Ar}(\mathcal{V}\mathsf{ect}^{\mathsf{inj}})$ classifies composite map of vector bundles over L_{pq}

$$\pi^*_{pq}TX_p \overset{D\pi^{\vee}_{pq}}{\hookrightarrow} TL_{pq} \overset{\text{zero}}{\hookrightarrow} TL_{pq}\oplus\epsilon^1 \overset{D\gamma_{pq}}{\hookrightarrow} \gamma^*_{pq}TX_q$$

with $D\pi^{\vee}_{pq}$ the dual of the derivative of π_{pq}, and with the middle map the inclusion as zero in the second coordinate.

Remark 2.17. In the above we made use of an identification $V \simeq V^{\vee}$ for each object in $\mathcal{V}\mathsf{ect}$. Such an identification is tantamount to a choice of an inner product on V, the space of which is contractible. It is because of these contractible choices that we opted for a hands-off construction of the constructible tangent bundle, which occupies §2.3. Indeed, we found it impractical to directly manage these contractible choices in order to define a functor between ∞-categories $\mathcal{E}\mathsf{xit} \xrightarrow{\mathsf{T}} \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$.

2.2. **Framings.** We introduce the essential geometric notion which is new to this article: *vari-framings*. Recall that a framing on a smooth n-manifold M is an identification $\epsilon^n_M \simeq \mathsf{T}_M$ of vector bundles over M. A vari-framing is a direct imitation of this classical notion. Recall from §2.1 the functors ϵ^{dim} and T from $\mathcal{E}\mathsf{xit}$ to $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$.

Definition 2.18. The space $\mathsf{vfr}(X)$ of *vari-framings* on a stratified space X is the space of lifts

$$\begin{tikzcd}
 & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \arrow[d, "\mathsf{diag}"] \\
\mathsf{Exit}(X) \arrow[ur, dashed, "\varphi"] \arrow[r, "(\epsilon^{\mathsf{dim}}_X, \mathsf{T}_X)"'] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \times \mathcal{V}\mathsf{ect}^{\mathsf{inj}} .
\end{tikzcd}$$

In particular, a vari-framing on a stratified space X is an equivalence of functors

$$\epsilon^{\mathsf{dim}}_X \underset{\varphi}{\simeq} \mathsf{T}_X \colon \mathsf{Exit}(X) \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}} .$$

Remark 2.19. The term 'vari-framing' is short for 'variform framing' which reflects that the framing varies over strata. See Remark 2.24.

Example 2.20. For each dimension n we give an example of a vari-framed stratified n-manifold: $\mathbb{D}^n$. We will reexamine this in §3.4 (see Definition 3.18). The underlying space of $\mathbb{D}^n$ is the unit disk in $\mathbb{R}^n$. Its stratifying poset $P^{\triangleright}$ is the right cone on the poset whose underlying set is $[n-1] \times \{\pm\}$ with partial order $(i,\sigma) \leq (i',\sigma')$ meaning $i < i'$ if $i \neq i'$ and otherwise $\sigma = \sigma'$. The stratification $\mathbb{D}^n \to P^{\triangleright}$ assigns to a vector $x = (x_1, \dots, x_n)$ the cone point if $\|x\| < 1$ and otherwise $(i_0, \mathsf{sign}(x_{i_0}))$ where $i_0 := \mathsf{Max}\{i \mid x_i \neq 0\}$. The conically smooth structure on this stratified space is inherited from the smooth structure of the closed n-disk.
Note that the projection $\mathsf{Exit}(\mathbb{D}^n) \xrightarrow{\simeq} P^{\triangleright}$ is an equivalence of ∞-categories. The constructible tangent bundle is the composition

$$\mathsf{T}_{\mathbb{D}^n} \colon \mathsf{Exit}(\mathbb{D}^n) \longrightarrow P^{\triangleright} \longrightarrow [n-1]^{\triangleright} = [n] \xrightarrow{i \mapsto \mathbb{R}^i} \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$$

that we now explain. The first arrow is the functor to the stratifying poset; the second arrow is the right cone on the projection $[n-1] \times \{\pm\} \xrightarrow{\mathsf{pr}} [n-1]$; and the third arrow is as indicated which carries each relation to the standard inclusions among Euclidean spaces. Presented in this way, there is a manifest vari-framing $\epsilon^{\mathsf{dim}}_{\mathbb{D}^n} \simeq \mathsf{T}_{\mathbb{D}^n}$.

We now introduce a notion of framing on a stratified space which is less sensitive to stratified structures.

Definition 2.21 (B-framings)**.** For each dimension n, and for each map of spaces $B \to \mathsf{BO}(n)$, the space of *B-framings* on a stratified space X is the space of lifts in the diagram among ∞-categories

$$\begin{tikzcd}
 & \mathsf{Ar}(\mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \underset{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}}{\times} B \arrow[r] \arrow[d] & B \arrow[d] \\
 & \mathsf{Ar}(\mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \arrow[r, "\mathsf{ev}_t"] \arrow[d, "\mathsf{ev}_s"] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \\
\mathsf{Exit}(X) \arrow[uur, dashed, "\varphi"] \arrow[r, "\mathsf{T}_X"] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} &
\end{tikzcd}$$

in which the right vertical arrow is the composite functor $B \to \mathsf{BO}(n) \hookrightarrow \mathcal{V}\mathsf{ect}^{\sim} \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$. The space

$$\mathsf{sfr}_n(X)$$

of *solid n-framings* is the special case that $B \simeq * \to \mathsf{BO}(n)$ is the inclusion of the base point. The space of *stratified n-manifold structures* is the special case that $B \xrightarrow{\simeq} \mathsf{BO}(n)$ is an equivalence.

2.2.1. **Explicating framings.** We explicate these notions of framings. We begin with the weakest of the structures above: stratified n-manifold structures.

Remark 2.22. Explicitly, a stratified n-manifold is a stratified space X together with a rank n vector bundle η on X, as well as an injection of constructible vector bundles $\mathsf{T}_X \hookrightarrow \eta$. By way of a zero-section, such data determines an embedding of X into the total space of the cokernel vector bundle: $X \hookrightarrow \eta/\mathsf{T}_X$. The codomain of this embedding is a topological manifold of dimension n (with regularity that we will not articulate). In this way, we regard $\mathsf{T}_X \hookrightarrow \eta$ as an infinitesimal thickening of X as a smooth n-manifold; in particular, the topological dimension of X is bounded above by n. Should X be a smooth n-manifold, then the injection $\mathsf{T}_X \hookrightarrow \eta$ is an isomorphism, and so η is no more data than that of X alone.

Remark 2.23. A B-structure on a stratified n-manifold $(X, \mathsf{T}_X \hookrightarrow \eta)$ is a lift of the classifying map $X \xrightarrow{\eta} \mathsf{BO}(n)$ to B. In particular, a solid framing on such a stratified n-manifold is an isomorphism of vector bundles $\eta \cong \epsilon^n_X$ with the trivial rank n vector bundle over X. In particular, the space of solid framings on such a stratified n-manifold is a torsor for $\mathsf{Map}\big(X, \mathsf{O}(n)\big)$, provided a solid framing exists; this depends only on the underlying space of X.

Remark 2.24. A vari-framing on a stratified n-manifold $(X, \mathsf{T}_X \hookrightarrow \eta)$ is a *trivialization* of $(\mathsf{T}_X \hookrightarrow \eta)$, by which we mean an equivalence of functors $\mathsf{Exit}(X) \to \mathsf{Ar}(\mathcal{V}\mathsf{ect}^{\mathsf{inj}})$

$$\big(\mathsf{T}_X \hookrightarrow \eta\big) \underset{\varphi}{\simeq} \big(\epsilon^{\mathsf{dim}}_X \hookrightarrow \epsilon^n_X\big) ~.$$

Here, $\mathsf{T}_X \colon \mathsf{Exit}(X) \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ is the functor defined in §2.1.4 while $\epsilon^{\mathsf{dim}}_X \colon \mathsf{Exit}(X) \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ is the functor whose restriction to the i-dimensional stratum X_i is the constant functor at $\mathbb{R}^i$, and whose value on the space of morphisms from X_i to X_j is the inclusion $\mathbb{R}^i \hookrightarrow \mathbb{R}^j$ as the first coordinates. In particular, a vari-framing on a stratified space X determines, for each dimension i, an equivalence of vector bundles on X_i

$$\mathsf{T}_{X_i} \underset{\varphi_i}{\simeq} \epsilon^i_{X_i} ~.$$

Also, a vari-framing on X determines, for each pair of dimensions $i \leq j$ with link system $X_i \xleftarrow{\pi_{ij}} L_{ij} \xrightarrow{\gamma_{ij}} X_j$, an identification of the projection

$$\epsilon^i_{L_{ij}} \underset{\varphi_i}{\simeq} \pi^*_{ij}\mathsf{T}_{X_i} \xleftarrow{D\pi_{ij}} \mathsf{T}_{L_{ij}} \xleftarrow{\mathsf{pr}} \mathsf{T}_{L_{ij}} \oplus \epsilon^1_{L_{ij}} \simeq (\mathsf{T}_{X_j})_{|L_{ij}} \underset{\varphi_{j|L_{ij}}}{\simeq} \epsilon^j_{L_{ij}}$$

with the standard projection off of the final coordinates. There is a likewise description for finite sequences of dimensions $i_0 \leq \cdots \leq i_p$.

Remark 2.25. From the description of link data in Remark 2.24, a vari-framing on a stratified space is not solely a framing of each stratum, but also coherent compatibility in terms of links of strata.

For M a smooth n-manifold, its space of solid framings

$$\mathsf{sfr}_n(M) \simeq \mathsf{Map}(M, \mathsf{O}(n))$$

is a torsor for the space of maps to the group of orthogonal transformations of $\mathbb{R}^n$, provided M admits a framing. We will give a likewise description of the space of vari-framings on a stratified space. We first introduce the stratified version of the orthogonal group $\mathsf{O}(n)$.

Construction 2.26 ($\underline{\mathsf{O}}$)**.** We define the ∞-category $\underline{\mathsf{O}}$ over the poset $\mathbb{Z}_{\geq 0}$ of non-negative integers, as the following complete Segal space. Space of functors $[p] \to \underline{\mathsf{O}}$ over a functor $[p] \xrightarrow{i_\bullet} \mathbb{Z}_{\geq 0}$ classifying $i_0 \leq \cdots \leq i_p$ is

$$\mathsf{O}(\mathbb{R}^{i_0} \subset \cdots \subset \mathbb{R}^{i_p}) \ \subset \ \mathsf{O}(\mathbb{R}^{i_p}) ~,$$

the underlying space of the smooth submanifold consisting of those orthogonal linear maps that preserve each standardly embedded $\mathbb{R}^{i_u} \subset \mathbb{R}^{i_p}$. A morphism $[p] \xrightarrow{\rho} [q] \xrightarrow{i_\bullet} \mathbb{Z}_{\geq 0}$ in $\mathbf{\Delta}_{/\mathbb{Z}_{\geq 0}}$ determines the smooth homomorphism $\mathsf{O}(\mathbb{R}^{i_0} \subset \cdots \subset \mathbb{R}^{i_q}) \to \mathsf{O}(\mathbb{R}^{i_{\rho(0)}} \subset \cdots \subset \mathbb{R}^{i_{\rho(p)}})$ given by restricting transformations from $\mathbb{R}^{i_q}$ to $\mathbb{R}^{i_{\rho(p)}} \subset \mathbb{R}^{i_q}$ and relaxing which subspaces are preserved. It follows that $\underline{\mathsf{O}}$ is indeed a simplicial space over $\mathbb{Z}_{\geq 0}$. For each $[p] \xrightarrow{i_\bullet} \mathbb{Z}_{\geq 0}$, taking orthogonal complements defines a diffeomorphism

$$\mathsf{O}(\mathbb{R}^{i_0} \subset \cdots \subset \mathbb{R}^{i_p}) \ \cong \ \mathsf{O}(\mathbb{R}^{i_0}) \times \prod_{0<k\leq p} \mathsf{O}(\mathbb{R}^{i_k - i_{k-1}}) ~.$$

This implies the simplicial space $\underline{\mathsf{O}}$ satisfies the Segal condition. By inspection, the map of simplicial spaces $\underline{\mathsf{O}} \to \mathbb{Z}_{\geq 0}$ has the property that only degenerate simplices are carried to degenerate simplices. Because the only equivalences of $\mathbb{Z}_{\geq 0}$ are identities, the Segal space $\underline{\mathsf{O}}$ satisfies the completeness condition. Therefore, $\underline{\mathsf{O}}$ presents an ∞-category over $\mathbb{Z}_{\geq 0}$. Finally, because it is the case value-wise as a simplicial space, this ∞-category is naturally a group object among ∞-categories over $\mathbb{Z}_{\geq 0}$.

Observation 2.27. For each stratified space X, a choice of vari-framing on X determines an identification of the space of vari-framings on X

$$\mathsf{vfr}(X) \ \simeq \ \mathsf{Map}_{/\mathbb{Z}_{\geq 0}}\big(\mathsf{Exit}(X), \underline{\mathsf{O}}\big)$$

with the space of functors over the poset $\mathbb{Z}_{\geq 0}$ from $\mathsf{Exit}(X) \xrightarrow{\mathsf{dim}} \mathbb{Z}_{\geq 0}$ to the ∞-category $\underline{\mathsf{O}}$ of Construction 2.26.

In the next result we denote the restriction $\underline{\mathsf{O}}_{|\leq n} := \{0 < \cdots < n\} \underset{\mathbb{Z}_{\geq 0}}{\times} \underline{\mathsf{O}}$, which is again a group object among ∞-categories.

Lemma 2.28. *For each dimension n, there is a morphism $\underline{\mathsf{O}}_{|\leq n} \to \mathsf{O}(n)$ between group objects among ∞-categories. This morphism induces an equivalence of group objects in spaces from the classifying space of the ∞-category $\underline{\mathsf{O}}_{|\leq n}$.*

Proof. For each $i_0 \leq \cdots \leq i_p \leq n$ there is a standard inclusion $\mathsf{O}(\mathbb{R}^{i_0} \subset \cdots \subset \mathbb{R}^{i_p}) \subset \mathsf{O}(n)$. For n fixed, these inclusions assemble as a functor $\underline{\mathsf{O}}_{|\leq n} \to \mathsf{O}(n)$, where here we regard $\mathsf{O}(n)$ as a the ∞-category associated to the underlying space of the smooth manifold $\mathsf{O}(n)$. Note the evident section $\mathsf{O}(n) \to \underline{\mathsf{O}}_{|\leq n}$. This section is in fact a left adjoint among ∞-categories, with the unit for the resulting adjunction given by extending linear maps on $\mathbb{R}^i$ to $\mathbb{R}^n$ as the identity on the final $(n-i)$ coordinates, for each $i \leq n$. In particular, this section is final.

□

Remark 2.29. For each stratified space X, there is a natural functor $\mathsf{Exit}(X) \to X$ to the underlying space. In §4 of [AFT] it is verified that this functor induces an equivalence $\mathsf{B}\,\mathsf{Exit}(X) \xrightarrow{\simeq} X$ between spaces. Suppose X admits a vari-framing and that its dimension is bounded above by n. Then, in particular, X admits a solid n-framing. Lemma 2.28 thus provides a map of spaces

$$\mathsf{vfr}(X) \simeq \mathsf{Map}_{/\mathbb{Z}_{\geq 0}}(\mathsf{Exit}(X), \underline{\mathsf{O}}) \longrightarrow \mathsf{Map}(X, \mathsf{O}(n)) \simeq \mathsf{sfr}_n(X) ~.$$

We now demonstrate the vast difference between these vari-framings and solid framings.

Remark 2.30. Consider the hemispherical disk $\mathbb{D}^3$ of Example 2.20. There is a fiber sequence of spaces

$$\Omega^3\mathsf{O}(3) \longrightarrow \mathsf{vfr}(\mathbb{D}^3) \longrightarrow \mathsf{vfr}(\partial\mathbb{D}^3)$$

in which the right arrow is given by restriction and the fiber is over the given vari-framing on $\partial\mathbb{D}^3$ In these low dimensions, the base of this fiber sequence is discrete, while this fiber has infinitely many connected components. On the other hand, the space $\mathsf{sfr}_3(\mathbb{D}^3) \simeq \mathsf{Map}\big(\mathbb{D}^3, \mathsf{O}(3)\big)$ has two components.

For each stratified space X, the assignment

$$X \underset{\mathsf{opn}}{\supset} U \ \mapsto\ \mathsf{vfr}(U)\ \in \mathcal{S}\mathsf{paces}$$

defines a constructible sheaf of spaces on X. In contrast with ordinary differential topology, where 'constructible' and 'locally constant' are the same notion, the existence of a vari-framing on a stratified space has strong global implications. We demonstrate this as the next observtion. To state it we use that a smooth n-manifold $\overline{M}$ with boundary defines a stratified space whose underlying topological space is that of $\overline{M}$, whose stratification $\overline{M} \to \{n-1 < n\}$ is such that the $(n-1)$-stratum is precisely the boundary, and whose conically smooth structure is inherited from the smooth structure of $\overline{M}$.

Lemma 2.31. *Let $\overline{M}$ be a compact connected smooth n-manifold with boundary whose boundary is not empty. Then a vari-framing on $\overline{M}$ determines a framed diffeomorphism*

$$\overline{M} \ \cong\ B \times \mathbb{D}^1\ ,$$

for some compact smooth framed $(n-1)$-manifold B.

Proof. Fix a vari-framing on $\overline{M}$. We find this diffeomorphism in two steps. First, the projection from the link $\mathsf{Link}_{\partial\overline{M}}(\overline{M}) \xrightarrow[\pi]{\cong} \partial\overline{M}$ is an isomorphism. The vari-framing gives an identification

$$\mathsf{T}_{\mathsf{Link}_{\partial\overline{M}}(\overline{M})} \oplus \epsilon^1_{\mathsf{Link}_{\partial\overline{M}}(\overline{M})} \ \simeq\ \epsilon^n_{\mathsf{Link}_{\partial\overline{M}}(\overline{M})}$$

under an identification $\pi^*\mathsf{T}_{\partial\overline{M}} \simeq \pi^*\epsilon^{n-1}_{\partial\overline{M}}$. Inspecting cokernels gives an automorphism of $\epsilon^1_{\mathsf{Link}_{\partial\overline{M}}(\overline{M})}$, which is the data of a map from the underlying space $\partial\overline{M} \underset{\pi}{\simeq} \mathsf{Link}_{\partial\overline{M}}(\overline{M}) \to \mathsf{O}(1) \simeq \{\pm\}$. This determines the desired partition.

Next, the first coordinate of the vari-framing defines a non-vanishing vector field on $\overline{M}$. Flowing along this vector field gives the partially defined smooth map $\overline{M} \times \mathbb{R} \dashrightarrow \overline{M}$ of manifolds with boundary. Reparametrizing, this flow restricts as the desired diffeomorphism

$$\partial_-\overline{M} \times [-1, 1] \ \cong\ \overline{M}\ .$$

That this isomorphism lifts as a vari-framed isomorphism is manifest from its construction.

$\square$

Remark 2.32. Lemma 2.31 is an instance of a general phenomenon. Namely, vari-framings are but one example of a local structure on stratified spaces. One could invent others: stipulate a structure stratum-wise, each of which could be quite different in nature, in addition to specified interactions among them over link data. For a given such stratified structure τ, the existence of a τ-structure on X will then restrict the global topology of X. In particular, the general obstruction to the existence of a τ-structure need not be measured by the cohomology of X, unless τ is locally constant.

2.3. **Characterization of the constructible tangent bundle.** Here we prove Proposition 2.14, which characterizes the constructible tangent bundle.

2.3.1. **Outline.** We outline the logic of the proof of Proposition 2.14. We seek a symmetric monoidal functor $\mathcal{E}\mathsf{xit} \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ under $\mathcal{V}\mathsf{ect}^{\sim}$ that is unique among all such that are local with respect to closed and embedding morphisms. Such a functor is equivalent to the data of a morphism $\mathsf{Map}([\bullet], \mathcal{E}\mathsf{xit}) \to \mathsf{Map}([\bullet], \mathcal{V}\mathsf{ect}^{\mathsf{inj}})$ between simplicial $\mathcal{E}_\infty$-spaces under $\mathsf{Map}([\bullet], \mathcal{V}\mathsf{ect}^{\sim}) \simeq \mathcal{V}\mathsf{ect}^{\sim}$ that carries embedding and closed simplices to degenerate simplices. We can swiftly accommodate these

localities using cospans. Namely, the morphism we seek is equivalent to a morphism of simplicial symmetric monoidal functors under $\mathcal{V}\mathsf{ect}^{\sim}$,

$$\widetilde{\mathsf{T}}\colon \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[\bullet]})^{\mathsf{cls\text{-}emb}} \to \mathsf{Map}([\bullet],\mathcal{V}\mathsf{ect}^{\mathsf{inj}}) ~.$$

This is because the target of this arrow lies in simplicial symmetric monoidal ∞-groupoids. Consequently, such an arrow factors through the value-wise classifying space with respect to the closed and embedding morphisms, thereby implementing the desired locality. We construct $\widetilde{\mathsf{T}}$, and verify it is unique, by induction on $\bullet$. The inductive step exploits the following parallel facts.

(1) As a $\mathcal{V}\mathsf{ect}^{\sim}$-module in spaces, $\mathsf{Map}([p],\mathcal{V}\mathsf{ect}^{\mathsf{inj}})$ is free on $\mathsf{Map}([p-1],\mathcal{V}\mathsf{ect}^{\mathsf{inj}})$. This is tantamount to our vector spaces being finite dimensional, and the zero vector space being initial in $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$.
(2) As a $\mathcal{V}\mathsf{ect}^{\sim}$-module in ∞-categories, $\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[p]})^{\mathsf{cls\text{-}emb}}$ receives a final functor from the free $\mathcal{V}\mathsf{ect}^{\sim}$-module on a certain ∞-subcategory of $\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[p]})^{\mathsf{cls\text{-}emb}}$. By virtue of $\widetilde{\mathsf{T}}$ being a morphism of $\mathcal{V}\mathsf{ect}^{\sim}$-modules over $\mathbb{Z}_{\geq 0}$, this ∞-subcategory must map to the generating space $\mathsf{Map}([p-1],\mathcal{V}\mathsf{ect}^{\mathsf{inj}})$ of the above fact (1). Therefore, the restriction of $\widetilde{\mathsf{T}}$ to this ∞-subcategory must factor through a standard face map to $\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[p-1]})^{\mathsf{cls\text{-}emb}}$.

The base case of this induction reveals the entirety of the construction of the constructible tangent bundle. Namely, for $[p]=[0]$ the map $\widetilde{\mathsf{T}}$ is a symmetric monoidal functor under $\mathcal{V}\mathsf{ect}^{\sim}$

$$\mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}} \longrightarrow \mathcal{V}\mathsf{ect}^{\sim} ~.$$

Because the target of this arrow is a symmetric monoidal ∞-groupoid, the existence and uniqueness of this arrow is equivalent to the functor $\mathcal{V}\mathsf{ect}^{\sim} \to \mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}}$ being a final functor. Because $\mathcal{V}\mathsf{ect}^{\sim}$ is an ∞-groupoid, this finality is the simple observation that the cospan of pointed stratified spaces

$$(x\in X) \xrightarrow{\mathsf{cls}} (x\in \overline{X}_p) \xleftarrow{\mathsf{emb}} (0\in T_xX_p)$$

represents a final object in cospans from $(x\in X)$ to vector spaces. Here, the first arrow is opposite of the inclusion $\overline{X}_p\subset X$ of the closure of the stratum $X_p\subset X$ in which x lies; the second arrow is an exponential map from the tangent space at x of the smooth manifold X_p.

2.3.2. **Cospans in $\mathcal{E}\mathsf{xit}$.** Recall from Definition 1.22 the ∞-subcategories

$$\mathcal{E}\mathsf{xit}^{\mathsf{cls}} ~\subset~ \mathcal{E}\mathsf{xit} ~\supset~ \mathcal{E}\mathsf{xit}^{\mathsf{emb}} ~,$$

each of which contains the equivalences. For each ∞-category $\mathcal{K}$, consider the ∞-category $\mathcal{E}\mathsf{xit}^{\mathcal{K}} := \mathsf{Fun}(\mathcal{K},\mathcal{E}\mathsf{xit})$ of functors. This ∞-category is equipped with a pair of ∞-subcategories

$$\mathsf{Fun}^{\mathsf{cls}}(\mathcal{K},\mathcal{E}\mathsf{xit}) ~\subset~ \mathcal{E}\mathsf{xit}^{\mathcal{K}} ~\supset~ \mathsf{Fun}^{\mathsf{emb}}(\mathcal{K},\mathcal{E}\mathsf{xit})$$

consisting of the same objects, which are functors $\mathcal{K}\to\mathcal{E}\mathsf{xit}$, and those natural transformations through closed/embedding morphisms. Using that constructible bundles pull back (see §6 of [AFR]), and that closed morphisms in $\mathcal{B}\mathsf{un}$ are opposites of proper constructible embeddings, this pair of ∞-subcategories satisfies Criterion 5.10 of §5.2. Through the conclusions of §5.2, cospans in $\mathcal{E}\mathsf{xit}^{\mathcal{K}}$ by closed and embedding morphisms organize as an ∞-category

$$\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{\mathcal{K}})^{\mathsf{cls\text{-}emb}} ~.$$

Because finite pullbacks preserve colimits in Strat, through Observation 5.11 we see that this ∞-category inherits a symmetric monoidal structure from that of $\mathcal{E}\mathsf{xit}$. Furthermore, there is a sequence of composable symmetric monoidal functors

$$\mathcal{V}\mathsf{ect}^{\sim} \xrightarrow{\mathsf{diag}} \mathsf{Map}(\mathcal{K},\mathcal{V}\mathsf{ect}^{\sim}) \longrightarrow \mathsf{Map}(\mathcal{K},\mathcal{E}\mathsf{xit}) \underset{\text{Obs } 5.12}{\longrightarrow} \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{\mathcal{K}})^{\mathsf{cls\text{-}emb}} ~. \tag{5}$$

The symmetric monoidal ∞-category $\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{\mathcal{K}})^{\mathsf{cls\text{-}emb}}$ under $\mathcal{V}\mathsf{ect}^{\sim}$ is contravariantly functorial in the argument $\mathcal{K}$, by construction. In particular, we obtain a simplicial symmetric monoidal ∞-category $\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[\bullet]})^{\mathsf{cls\text{-}emb}}$ under $\mathcal{V}\mathsf{ect}^{\sim}$.

For $\mathcal{K}$ an ∞-category, consider the full symmetric monoidal ∞-subcategory

$$\mathsf{cSpan}\big(\mathcal{E}\mathsf{xit}^{\mathcal{K}^{\triangleleft}}\big)_0^{\mathsf{cls\text{-}emb}} \ \subset\ \mathsf{cSpan}\big(\mathcal{E}\mathsf{xit}^{\mathcal{K}^{\triangleleft}}\big)^{\mathsf{cls\text{-}emb}}$$

consisting of those functors $\mathcal{K}^{\triangleleft} \to \mathcal{E}\mathsf{xit}$ whose value on the cone-point $* \to \mathcal{E}\mathsf{xit}$ classifies a pointed stratified space $(x \in X)$ that is isomorphic to an open cone on some compact stratified space $(* \in \mathsf{C}(L))$ equipped with its cone-point.

The ∞-category $\mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}}$ is designed for the following technical result. This result articulates a sense in which the symmetric monoidal ∞-groupoid $\mathcal{V}\mathsf{ect}^{\sim}$, which classifies vector bundles, approximates $\mathcal{E}\mathsf{xit}$. The basic geometric idea is that each pointed stratified space $(x \in X)$ admits a canonical cospan in $\mathcal{E}\mathsf{xit}$ to $\mathsf{T}_x X_p$, the tangent space at x of the stratum $X_p \subset X$ in which x lies.

Lemma 2.33. *For each compact stratified space Z, the restriction of the symmetric monoidal structure*

$$\mathcal{V}\mathsf{ect}^{\sim} \times \mathsf{cSpan}\big(\mathcal{E}\mathsf{xit}^{\mathsf{Exit}(\overline{\mathsf{C}}(Z))}\big)_0^{\mathsf{cls\text{-}emb}} \xrightarrow{\ \times\ } \mathsf{cSpan}\big(\mathcal{E}\mathsf{xit}^{\mathsf{Exit}(\overline{\mathsf{C}}(Z))}\big)^{\mathsf{cls\text{-}emb}}$$

defines a final functor. In the case $Z = \emptyset$, the functor

$$\mathcal{V}\mathsf{ect}^{\sim} \longrightarrow \mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}} .$$

is a final functor.

Proof. Using Quillen's Theorem A, we show that for each object $\big(X \overset{\sigma}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big)$ of $\mathsf{cSpan}\big(\mathcal{E}\mathsf{xit}^{\mathsf{Exit}(\overline{\mathsf{C}}(Z))}\big)_0^{\mathsf{cls\text{-}emb}}$, the classifying space of the under ∞-category

$$\mathsf{B}\Big(\mathcal{V}\mathsf{ect}^{\sim} \times \mathsf{cSpan}\big(\mathcal{E}\mathsf{xit}^{\mathsf{Exit}(\overline{\mathsf{C}}(Z))}\big)_0^{\mathsf{cls\text{-}emb}}\Big)^{(X \overset{\sigma}{\leftrightarrows} \overline{\mathsf{C}}(Z))/} \ \simeq\ *$$

is terminal. We do this by demonstrating a terminal object in this under ∞-category. Manifestly, the ∞-category $\mathsf{cSpan}\big(\mathcal{E}\mathsf{xit}^{\mathsf{Exit}(\overline{\mathsf{C}}(Z))}\big)^{\mathsf{cls\text{-}emb}}$ has a factorization system whose left factor is *closed* morphisms and whose right factor is *embedding* morphisms. Therefore, it is sufficient to use the following logic.

(1) Demonstrate a terminal object

$$\big(X \overset{\sigma}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big) \xrightarrow{\ \mathsf{cls}\ } \big(\overline{X} \overset{\overline{\sigma}}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big)$$

of the under ∞-category $\mathsf{Fun}^{\mathsf{cls}}\Big(\mathsf{Exit}(\overline{\mathsf{C}}(Z)), \mathcal{E}\mathsf{xit}\Big)^{(X \overset{\sigma}{\leftrightarrows} \overline{\mathsf{C}}(Z))/}$.

(2) Demonstrate an initial object

$$\big(V \times R \overset{0 \times \tau}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big) \xrightarrow{\ \mathsf{emb}\ } \big(\overline{X} \overset{\overline{\sigma}}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big)$$

of the over ∞-category $\Big(\mathcal{V}\mathsf{ect}^{\sim} \times \mathsf{Fun}_0^{\mathsf{emb}}\big(\mathsf{Exit}(\overline{\mathsf{C}}(Z)), \mathcal{E}\mathsf{xit}\big)\Big)_{/(\overline{X} \overset{\overline{\sigma}}{\leftrightarrows} \overline{\mathsf{C}}(Z))}$. Here the subscript 0 indicates the full ∞-subcategory of those functors that carry the cone-point to a pointed stratified space $(\sigma(*) \in R_{|*})$ for which $\sigma(*)$ is the unique 0-dimensional stratum.

The first terminal object $\big(\overline{X} \overset{\overline{\sigma}}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big)$ can be described as follows. As a properly embedded constructible subspace, $\overline{X} \subset X$ is the intersection of all such that contain the image of the section σ. The projection $\overline{X} \to \overline{\mathsf{C}}(Z)$ is a constructible bundle; this constructible bundle is equipped with a section over the section σ, by construction. Also by construction, there is a canonical closed morphism $\big(X \overset{\sigma}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big) \to \big(\overline{X} \overset{\overline{\sigma}}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big)$ in the functor ∞-category $\mathsf{Fun}\big(\mathsf{Exit}(\overline{\mathsf{C}}(Z)), \mathcal{E}\mathsf{xit}\big)$. The construction of $\overline{X}$ makes this closed morphism manifestly terminal among all such.

So we can assume that the canonical closed morphism $X \to \overline{X}$ is an equivalence. We now face the problem of showing there is an initial embedding morphism $\big(V \times R \overset{\tau}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big) \xrightarrow{\ \mathsf{emb}\ } \big(X \overset{\overline{\sigma}}{\leftrightarrows} \overline{\mathsf{C}}(Z)\big)$ between functors $\mathsf{Exit}(\overline{\mathsf{C}}(Z)) \to \mathcal{E}\mathsf{xit}$. Using Lemma 2.34, it is sufficient to argue the existence of

an initial object of the over ∞-category $(\mathcal{S}\mathsf{trat}^{*/})^{\mathsf{emb}}{}_{/(x\in X)}$. From the very definition of a stratified space in the sense of §3 of [AFT], there is a basic neighborhood $\bigl((0,*)\in\mathbb{R}^i\times\mathsf{C}(L)\bigr)\hookrightarrow(x\in X)$, and such basic neighborhoods form a basis for the topology about $x\in X$. This implies the full ∞-subcategory consisting of basic neighborhoods is initial. In §4 of [AFT] it is shown that this ∞-subcategory is in fact a contractible ∞-groupoid. This verifies the desired initiality.

We now verify the second clause of the lemma. We continue with the same notation above. In the case $Z=\emptyset$, the object $\bigl(X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z)\bigr)$ is simply a pointed stratified space $(x\in X)$. As so, $\overline{X}$ is simply the closure $\overline{X}_p\subset X$ of the stratum $x\in X_p\subset X$ in which x lies. In particular, x lies in the top-dimensional stratum of the stratified space $\overline{X}_p$. Therefore, the object $R=*$ is a point. Thus, the under ∞-category $(\mathcal{V}\mathsf{ect}^{\sim})^{(x\in X)/}$ has a terminal object; in particular its classifying space is terminal. Through Quillen's Theorem A, this proves that the functor $\mathcal{V}\mathsf{ect}^{\sim}\to\mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls}\text{-}\mathsf{emb}}$ is final.

$\square$

In the proof of Lemma 2.33 above, we made use of the following technical result. We denote by
$$(\mathcal{S}\mathsf{trat}^{*/})^{\mathsf{emb}}$$
the ∞-category of pointed stratified spaces and pointed open embeddings among them.

Lemma 2.34. *Let Z be a compact stratified space. Let $\mathsf{Exit}(\overline{\mathsf{C}}(Z))\to\mathcal{E}\mathsf{xit}$ be a functor classifying a constructible vector bundle $X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z)$ equipped with a section. Evaluation at the cone-point*
$$\Bigl(\mathcal{V}\mathsf{ect}^{\sim}\times\mathsf{Fun}_0^{\mathsf{emb}}\bigl(\mathsf{Exit}(\overline{\mathsf{C}}(Z)),\mathcal{E}\mathsf{xit}\bigr)\Bigr)_{/(X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z))}\longrightarrow(\mathcal{S}\mathsf{trat}^{*/})^{\mathsf{emb}}{}_{/(\sigma(*)\in X_{|*})}$$
is a right adjoint.

Proof. In §6 of [AFR] we construct a colimit diagram among stratified spaces over $\overline{\mathsf{C}}(Z)$:
$$\begin{tikzcd}
& \mathsf{Link}_{X_{|*|}}(X)\times Z\times(\Delta^1\smallsetminus\Delta^{\{0\}}) \arrow[r, "\gamma"] \arrow[d] & X_{|Z}\times(\Delta^1\smallsetminus\Delta^{\{0\}}) \arrow[dd] \\
\mathsf{Link}_{X_{|*|}}(X) \arrow[r, "\{0\}"] \arrow[d, "\pi"] & \mathsf{Link}_{X_{|*|}}(X)\times Z\times\Delta^1 \arrow[dr] & \\
X_{|*} \arrow[rr] & & X
\end{tikzcd}$$
in which the map π is proper and constructible and the map γ is open. Such a diagram is determined upon a choice of collaring of the above link over the standard collaring $Z\times\Delta^1\to\overline{\mathsf{C}}(Z)$. Because such collarings form a basis for the topology about $X_{|*}\subset X$, the compactness of Z grants the existence of $0<\epsilon\leq 1$ for which the image of the restricted section lies in the image of γ:
$$\mathsf{Image}\bigl(\sigma_{|Z\times(0,\epsilon)}\bigr)\ \subset\ \mathsf{Image}\bigl(\gamma_{|\,\mathsf{Link}_{X_{|*|}}(X)\times Z\times(0,\epsilon)}\bigr)\ .$$
(In the above expression we have made use of a standard identification $\Delta^1\cong[0,1]$ that carries $\Delta^{\{0\}}$ to $\{0\}$.) Therefore, such collarings can be chosen to guarantee that ϵ can be taken to be 1:
$$\mathsf{Image}(\sigma_{|Z\times(\Delta^1\smallsetminus\Delta^{\{0\}})})\ \subset\ \mathsf{Image}(\gamma)\ .$$
We thusly obtain a functor
$$\gamma_*\pi^*\colon(\mathcal{S}\mathsf{trat}^{*/})^{\mathsf{emb}}{}_{/(\sigma(*)\in X_{|*})}\longrightarrow\mathsf{Fun}^{\mathsf{emb}}\bigl(\mathsf{Exit}(\overline{\mathsf{C}}(Z)),\mathcal{E}\mathsf{xit}\bigr)_{/\left(X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z)\right)}$$
given by pullback along π and pushforward along γ; in terms of transversality sheaves this is the assignment
$$X_{|*}\underset{\rm open}{\supset}U\ \mapsto\ \Bigl(U\coprod_{\pi^{-1}(U)}\pi^{-1}(U)\times Z\times\Delta^1\coprod_{\pi^{-1}(U)\times Z\times(\Delta^1\smallsetminus\Delta^{\{0\}})}\gamma\bigl(\pi^{-1}(U)\times Z\times(\Delta^1\smallsetminus\Delta^{\{0\}})\bigr)\Bigr)\underset{\rm open}{\subset}X\ .$$

We now argue that this functor $\gamma_\ast\pi^\ast$ is a left adjoint to evaluation at the cone-point.

By inspection, there is a canonical identification of the composite functor

$$(\mathcal{S}\mathsf{trat}^{\ast/})^{\mathsf{emb}}{}_{/(\sigma(\ast)\in X_{|\ast})} \xrightarrow{\gamma_\ast\pi^\ast} \mathsf{Fun}^{\mathsf{emb}}\bigl(\mathsf{Exit}(\overline{\mathsf{C}}(Z)),\mathcal{E}\mathsf{xit}\bigr)_{/\left(X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z)\right)} \xrightarrow{\mathsf{ev}_\ast} (\mathcal{S}\mathsf{trat}^{\ast/})^{\mathsf{emb}}{}_{/(\sigma(\ast)\in X_{|\ast})}$$

with the identity functor. It remains to construct a counit transformation $\gamma_\ast\pi^\ast\circ\mathsf{ev}_\ast\to\mathsf{id}$. Consider an object $\bigl(W\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z)\bigr)\xrightarrow{\mathsf{emb}}\bigl(X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z)\bigr)$ of $\mathsf{Fun}^{\mathsf{emb}}\bigl(\mathsf{Exit}(\overline{\mathsf{C}}(Z)),\mathcal{E}\mathsf{xit}\bigr)_{/\left(X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z)\right)}$. Evaluation at the cone-point determines the map between morphism spaces

$$\mathsf{Map}_{/(X\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z))}\Bigl(\gamma_\ast\pi^\ast(\sigma(\ast)\in U),(W\overset{\sigma}{\leftrightarrows}\overline{\mathsf{C}}(Z))\Bigr)\longrightarrow\mathsf{Map}_{/(\sigma(\ast)\in X_{|\ast})}\Bigl((\sigma(\ast)\in U),(x\in W_{|\ast})\Bigr)~.$$

The fiber F_f of this map over a pointed open embedding $f\colon U\hookrightarrow W_{|\ast}$ is the space of extensions of f to open embeddings

$$\gamma_\ast\pi^\ast(U)\longrightarrow W$$

over and under $\overline{\mathsf{C}}(Z)$. We must show that this fiber space F_f is contractible. First, note that it is clearly nonempty.

For K a compact stratified space, consider a K-point $K\to F_f$. This consists of a conically smooth open embedding

$$\widetilde{f}\colon\gamma_\ast\pi^\ast(U)\times K~\hookrightarrow~W\times K$$

over and under $\overline{\mathsf{C}}(Z)\times K$, together with an identification of the restriction $\widetilde{f}_{|U\times K}\simeq f\times\mathsf{id}_K$. First, recall that collar-neighborhoods of $X_{|\ast}\subset X$ over collar-neighborhoods of $\ast\in\overline{\mathsf{C}}(Z)$ form a basis for the topology about $X_{|\ast}$. From this, together with compactness of K and of Z, there exists a conically smooth map $\epsilon\colon X_{|\ast}\to(0,1]$. This map has the property that, for each compact subspace $C\subset X_{|\ast}$, the subspace $\gamma\bigl(\pi^{-1}(C)\times[0,\epsilon(x)]\bigr)\subset X$ lies in the image of $\widetilde{f}$. By the construction of $\gamma_\ast\pi^\ast$ in terms of collarings, we conclude that the map $\widetilde{f}$ extends to an open embedding

$$\overline{\widetilde{f}}\colon\gamma_\ast\pi^\ast(U)\times\mathsf{C}(K)~\hookrightarrow~W\times\mathsf{C}(K)$$

over and under $\overline{\mathsf{C}}(Z)\times\mathsf{C}(K)$, together with an identification of the restriction $\overline{\widetilde{f}}_{|U\times\mathsf{C}(K)}\simeq f\times\mathsf{id}_{\mathsf{C}(K)}$. Therefore our K-point $K\to F_f$ is null-homotopic. This verifies the contractibility of the space F_f. □

2.3.3. **Proof of Proposition 2.14.** There is a sequence of fully faithful functors from the ∞-category of symmetric monoidal ∞-categories

$$\mathsf{Alg}_{\mathsf{Com}}(\mathsf{Cat}_\infty^\times)\hookrightarrow\mathsf{Alg}_{\mathsf{Com}}\bigl(\mathsf{PShv}(\mathbf{\Delta})^\times\bigr)\simeq\mathsf{Fun}\bigl(\mathbf{\Delta}^{\mathsf{op}},\mathsf{Alg}_{\mathsf{Com}}(\mathcal{S}\mathsf{paces}^\times)\bigr)\hookrightarrow\mathsf{Fun}\bigl(\mathbf{\Delta}^{\mathsf{op}},\mathsf{Alg}_{\mathsf{Com}}(\mathsf{Cat}_\infty^\times)\bigr)$$

to the ∞-category of simplicial symmetric monoidal ∞-categories, as we now explain. The first functor is induced by the presentation $\mathsf{Cat}_\infty\subset\mathsf{PShv}(\mathbf{\Delta})$ as complete Segal spaces, which is fully faithful and preserves finite products. The middle equivalence is an adjunction of variables, using that the Cartesian symmetric monoidal structure of presheaf ∞-categories is given value-wise. The last functor is induced from the fully faithful inclusion $\mathcal{S}\mathsf{paces}\hookrightarrow\mathsf{Cat}_\infty$ as ∞-groupoids, which preserves finite products.

In light of this fully faithful functor, it is enough to argue the existence of T as a morphism between simplicial symmetric monoidal ∞-categories under $\mathcal{V}\mathsf{ect}^\sim\simeq\mathsf{Map}([\bullet],\mathcal{V}\mathsf{ect}^\sim)$,

$$\mathsf{T}\colon~\mathsf{Map}\bigl([\bullet],\mathcal{E}\mathsf{xit}\bigr)\longrightarrow\mathsf{Map}\bigl([\bullet],\mathcal{V}\mathsf{ect}^{\mathsf{inj}}\bigr)~,$$

and argue that it is the unique such whose values on the closed and embedding simplices factor through degenerate simplices. The sequence (5) gives a morphism of simplicial symmetric monoidal ∞-categories $\mathsf{Map}([\bullet],\mathcal{E}\mathsf{xit})\to\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[\bullet]})^{\mathsf{cls\text{-}emb}}$. Of course, for each $p\geq0$ the symmetric monoidal ∞-category $\mathsf{Map}\bigl([p],\mathcal{V}\mathsf{ect}^{\mathsf{inj}}\bigr)$ is actually a symmetric monoidal ∞-groupoid. Thus,

in light of this alleged locality of T, it is equivalent to argue the existence and uniqueness of a morphism of simplicial symmetric monoidal ∞-categories under $\mathcal{V}\mathsf{ect}^{\sim}$,

$$\widetilde{\mathsf{T}}\colon \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[\bullet]})^{\mathsf{cls\text{-}emb}} \longrightarrow \mathsf{Map}([\bullet], \mathcal{V}\mathsf{ect}^{\mathsf{inj}})\ . \tag{6}$$

For $p \geq 0$, consider the full subcategory $\mathbf{\Delta}_{\leq p} \subset \mathbf{\Delta}$ consisting of those $[q]$ for which $q \leq p$. The canonical functor $\underset{p\geq 0}{\mathsf{colim}}\mathbf{\Delta}_{\leq p} \to \mathbf{\Delta}$ is an equivalence. It follows that the canonical functor $\mathsf{Fun}\big(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Alg}_{\mathsf{Com}}(\mathsf{Cat}_\infty^\times)\big) \to \lim_{p\geq 0} \mathsf{Fun}\big(\mathbf{\Delta}_{\leq p}, \mathsf{Alg}_{\mathsf{Com}}(\mathsf{Cat}_\infty^\times)\big)$ is again an equivalence. Therefore, to argue the existence and uniqueness of the morphism of simplicial symmetric monoidal ∞-categories (6) under $\mathcal{V}\mathsf{ect}^{\sim}$ it is enough to argue the existence and uniqueness of a morphism of truncated simplicial symmetric monoidal ∞-categories under $\mathcal{V}\mathsf{ect}^{\sim}$

$$\widetilde{\mathsf{T}}_{|\mathbf{\Delta}_{\leq p}}\colon\ \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[\bullet]})^{\mathsf{cls\text{-}emb}}_{|\mathbf{\Delta}_{\leq p}} \longrightarrow \mathsf{Map}([\bullet], \mathcal{V}\mathsf{ect}^{\mathsf{inj}})_{|\mathbf{\Delta}_{\leq p}} \tag{7}$$

for each $p \geq 0$, coherently compatibly. We do this by induction on $p \geq 0$.

The base case of $p = 0$ is the assertion that there is a unique symmetric monoidal retraction $\widetilde{\mathsf{T}}_{\{[0]\}}\colon\ \mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}} \to \mathcal{V}\mathsf{ect}^{\sim}$. Because $\mathcal{V}\mathsf{ect}^{\sim}$ is in particular an ∞-groupoid, this is equivalent to the assertion that the functor $\mathcal{V}\mathsf{ect}^{\sim} \to \mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}}$ induces an equivalence on classifying spaces:

$$\mathcal{V}\mathsf{ect}^{\sim} \xrightarrow{\simeq} \mathsf{B}\big(\mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}}\big)\ .$$

This is implied by the second clause of Lemma 2.33, which states that the functor $\mathcal{V}\mathsf{ect}^{\sim} \to \mathsf{cSpan}(\mathcal{E}\mathsf{xit})^{\mathsf{cls\text{-}emb}}$ is final.

We proceed by induction and assume that the morphism (7) has been defind, and been verified as being unique, for $0 \leq p < q$, coherently compatibly with the simplicial morphisms in $\mathbf{\Delta}_{\leq p}$. We must argue that there is a unique symmetric monoidal functor under $\mathcal{V}\mathsf{ect}^{\sim}$

$$\widetilde{\mathsf{T}}_{\{[q]\}}\colon\ \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[q]})^{\mathsf{cls\text{-}emb}} \longrightarrow \mathsf{Map}([q], \mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \tag{8}$$

for which, for each simplicial morphism $[p] \xrightarrow{\rho} [q]$ with $p < q$, the restriction $\rho^*\widetilde{\mathsf{T}}_{\{[q]\}}$ is coherently identified with $\widetilde{\mathsf{T}}_{\{[p]\}}$. For $\mathcal{K}$ an ∞-category, consider the full symmetric monoidal ∞-subcategory

$$\mathsf{Map}_0(\mathcal{K}^{\triangleleft}, \mathcal{V}\mathsf{ect}^{\mathsf{inj}})\ \subset\ \mathsf{Map}(\mathcal{K}^{\triangleleft}, \mathcal{V}\mathsf{ect}^{\mathsf{inj}})$$

consisting of those functors whose value on the cone-point $*$ is the zero vector space. The requirement of an identification of the restriction $(\widetilde{\mathsf{T}}_{\{[q]\}})_{|\{[0]\}}$ with $\widetilde{\mathsf{T}}_{\{[0]\}}$ implies that $\widetilde{\mathsf{T}}_{\{[q]\}}$ restricts as a morphism between these symmetric monoidal ∞-subcategories under $\mathcal{V}\mathsf{ect}^{\sim}$

$$\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[q]})^{\mathsf{cls\text{-}emb}}_0 \longrightarrow \mathsf{Map}_0([q], \mathcal{V}\mathsf{ect}^{\mathsf{inj}})\ .$$

The requirement that $\widetilde{\mathsf{T}}_{\{[q]\}}$ be symmetric monoidal under $\mathcal{V}\mathsf{ect}^{\sim}$ in particular requires a commutative diagram among ∞-categories

$$\begin{tikzcd}
\mathcal{V}\mathsf{ect}^{\sim} \times \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[q]})^{\mathsf{cls\text{-}emb}}_0 \arrow[r, "\times", "(\mathrm{f})"'] \arrow[d, "\mathsf{id}\times\widetilde{\mathsf{T}}_{\{[q]\}}"'] & \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[q]})^{\mathsf{cls\text{-}emb}} \arrow[d, "\widetilde{\mathsf{T}}_{\{[q]\}}"] \\
\mathcal{V}\mathsf{ect}^{\sim} \times \mathsf{Map}_0([q], \mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \arrow[r, "\oplus"] & \mathsf{Map}([q], \mathcal{V}\mathsf{ect}^{\mathsf{inj}}).
\end{tikzcd}$$

Lemma 2.33, applied to $Z = \Delta^{q-1}$, states that the functor labeled as (f) is final. Therefore, since $\mathsf{Map}([q], \mathcal{V}\mathsf{ect}^{\mathsf{inj}})$ is in particular an ∞-groupoid, the existence and uniqueness of $\widetilde{\mathsf{T}}_{\{[q]\}}$ is equivalent to the existence and uniqueness of its restriction $\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[q]})^{\mathsf{cls\text{-}emb}}_0 \to \mathsf{Map}_0([q], \mathcal{V}\mathsf{ect}^{\mathsf{inj}})$.

We have required that restriction along the standard simplicial morphism $[q-1] = \{1 < \cdots < q\} \xrightarrow{\rho} [q]$ determines a commutative diagram among ∞-categories

$$\begin{tikzcd}
\mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[q]})_0^{\mathsf{cls\text{-}emb}} \arrow[r, "\rho^*"] \arrow[d, "\widetilde{\mathsf{T}}_{\{[q]\}}"'] & \mathsf{cSpan}(\mathcal{E}\mathsf{xit}^{[q-1]})^{\mathsf{cls\text{-}emb}} \arrow[d, "\widetilde{\mathsf{T}}_{\{[q-1]\}}"] \\
\mathsf{Map}_0([q], \mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \arrow[r, "\rho^*", "(\mathrm{z})"'] & \mathsf{Map}([q-1], \mathcal{V}\mathsf{ect}^{\mathsf{inj}})
\end{tikzcd} .$$

Because the zero vector space is initial in the ∞-category $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$, the bottom horizontal arrow (z) is an equivalence of ∞-groupoids. Thus, the existence and uniqueness of the functor $\widetilde{\mathsf{T}}_{\{[q]\}}$, which is the left vertical arrow in the above diagram, is implied by the existence and uniqueness of $\widetilde{\mathsf{T}}_{\{[q-1]\}}$, which is the right vertical arrow in the above diagram. Our induction hypothesis on q ensures the existence and uniqueness of $\widetilde{\mathsf{T}}_{\{[q-1]\}}$, coherently compatibly with the simplicial morphisms in $\mathbf{\Delta}_{<q}$. Through this logic we conclude the existence and uniqueness of $\widetilde{\mathsf{T}}_{\{[q]\}}$ as in (8). By construction, this symmetric monoidal functor is coherently compatible with the simplicial morphisms in $\mathbf{\Delta}_{\leq q}$, thereby completing the inductive step.

2.4. **Tangential structures.** In the previous section we introduced a variety of structures on one stratified space at a time. We are interested in constructible families of such structured stratified spaces. In this section we introduce a general notion of *tangential structures* among stratified spaces so as to accommodate several expected functorialities.

We concern ourselves with placing structures fiberwise on constructible families of stratified spaces. Toward phrasing what such data is, we first consider how to phrase such for a single stratified space. Along the lines of the main results of [AFT], such a structure can be taken to be a functor from its exit-path ∞-category. And so, given a constructible bundle $X \to K$, we will take a fiberwise structure on X to be a functor from its exit-path ∞-category which is constant in the K-direction. This is precisely what the ∞-category $\mathcal{E}\mathsf{xit}$ over $\mathcal{B}\mathsf{un}$ accommodates, as the next definition articulates.

Recall Definition 1.22 of a closed morphism in $\mathcal{E}\mathsf{xit}$.

Definition 2.35. A *tangential structure* is an ∞-category τ equipped with a functor $\tau \to \mathcal{E}\mathsf{xit}$ that satisfies the following condition.

> Let $[1] \xrightarrow{f} \mathcal{E}\mathsf{xit}$ classify a closed morphism. Then each solid diagram among ∞-categories
>
> $$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \tau \arrow[d] \\
{[1]} \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & \mathcal{E}\mathsf{xit}
\end{tikzcd} ,$$
>
> admits a filler that classifies a coCartesian morphism over $\mathcal{E}\mathsf{xit}$. Furthermore, the functor between fiber ∞-categories $f_!\colon \tau_{|f(0)} \to \tau_{|f(1)}$ is an equivalence.

Such a morphism $\widetilde{f}$ of τ is a *closed-coCartesian* morphism: it is a morphism in τ which is coCartesian over $\mathcal{E}\mathsf{xit}$ whose image in $\mathcal{E}\mathsf{xit}$ is a closed morphism. The ∞-category of *tangential structures* is the ∞-subcategory of $\mathsf{Cat}_{\infty/\mathcal{E}\mathsf{xit}}$ whose objects are tangential structures and whose morphisms are those functors $\tau \to \tau'$ over $\mathcal{E}\mathsf{xit}$ that carry closed-coCartesian morphisms to closed-coCartesian morphisms.

We typically denote a tangential structure $\tau \to \mathcal{E}\mathsf{xit}$ simply as τ with the functor to $\mathcal{E}\mathsf{xit}$ understood.

Observation 2.36. The monomorphism from tangential structures into $\mathsf{Cat}_{\infty/\mathcal{E}\mathsf{xit}}$ preserves and detects limits. This is to say, a limit diagram $\mathcal{J}^{\triangleleft} \to (\mathsf{Cat}_{\infty})_{/\mathcal{E}\mathsf{xit}}$ factors through tangential structures

whenever the restriction $\mathcal{J} \to (\mathsf{Cat}_\infty)_{/\,\mathcal{E}\mathsf{xit}}$ does. In particular, for $\tau \to \mathcal{E}\mathsf{xit}$ and $\tau' \to \mathcal{E}\mathsf{xit}$ tangential structures, then the canonical functor $\tau \underset{\mathcal{E}\mathsf{xit}}{\times} \tau' \to \mathcal{E}\mathsf{xit}$ too is a tangential structure.

Observation 2.37. After Observation 2.36, through presentability considerations there is a left adjoint to the inclusion from tangential structures into $\mathsf{Cat}_{\infty/\,\mathcal{E}\mathsf{xit}}$. In particular, each ∞-category over $\mathcal{E}\mathsf{xit}$ naturally determines a tangential structure.

We temporarily denote the forgetful functor $\mathcal{E}\mathsf{xit} \xrightarrow{\pi} \mathcal{B}\mathsf{un}$ given by forgetting section data. Lemma 5.20 of the appendix verifies that the base change functor

$$\pi^*\colon \mathsf{Cat}_{\infty/\,\mathcal{B}\mathsf{un}} \rightleftarrows \mathsf{Cat}_{\infty/\,\mathcal{E}\mathsf{xit}} \colon \pi_* ~,$$

has a right adjoint, as depicted.

Definition 2.38 ($\mathcal{B}\mathsf{un}^\tau$)**.** For each tangential structure $\tau \to \mathcal{E}\mathsf{xit}$, the ∞-category of *τ-structured stratified spaces* is the ∞-subcategory

$$\mathcal{B}\mathsf{un}^\tau \hookrightarrow \pi_*\tau$$

over $\mathcal{B}\mathsf{un}$ for which a functor $\mathcal{K} \to \pi_*\tau$ factors through $\mathcal{B}\mathsf{un}^\tau$ if its adjoint functor $\mathcal{E}\mathsf{xit}_{|\mathcal{K}} \to \tau$ over $\mathcal{E}\mathsf{xit}$ carries closed-coCartesian morphisms to coCartesian morphisms. For each class ψ of morphisms of $\mathcal{B}\mathsf{un}$, the ∞-subcategory

$$\mathcal{B}\mathsf{un}^{\tau,\psi} := \mathcal{B}\mathsf{un}^\psi \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^\tau \subset \mathcal{B}\mathsf{un}^\tau$$

which is the pullback over $\mathcal{B}\mathsf{un}^\psi$.

Remark 2.39. Explicitly, a τ-structured stratified space is a stratified space X together with a lift

$$\begin{tikzcd} & \tau \arrow[d] \\ \mathsf{Exit}(X) \arrow[r] \arrow[ur, dashed, "g"] & \mathcal{E}\mathsf{xit}\,. \end{tikzcd}$$

Should the pullback $\tau_{|X} \to \mathsf{Exit}(X)$ be the unstraightening of a functor $\mathsf{Exit}(X) \xrightarrow{\tau_{|X}} \mathcal{S}\mathsf{paces}$, such a lift is the datum of a point $g \in \lim\big(\mathsf{Exit}(X) \xrightarrow{\tau_{|X}} \mathcal{S}\mathsf{paces}\big)$ in the limit. Through Theorem A.9.3 of [Lu2], we interpret such a g as a global section of a constructible sheaf on the stratified space X. In this way, we think of the notaion of a tangential structure as an expansion of that of a constructible sheaf on the site of stratified spaces and open embeddings among them.

Observation 2.40. By design, for each tangential structure τ, the projection $\mathcal{B}\mathsf{un}^\tau \to \mathcal{B}\mathsf{un}$ preserves and detects pullback diagrams that factor through $\mathcal{B}\mathsf{un}^{\tau,\mathsf{cls}}$.

Lemma 2.41. *For each tangential structure τ, the pair of ∞-subcategories* $(\mathcal{B}\mathsf{un}^{\tau,\mathsf{cls}}, \mathcal{B}\mathsf{un}^{\tau,\mathsf{act}})$ *is a factorization system on* $\mathcal{B}\mathsf{un}^\tau$*.*

Proof. In §6 of [AFR] we prove the case in which $\tau \to \mathcal{E}\mathsf{xit}$ is an equivalence. The result follows because $\mathcal{B}\mathsf{un}^\tau \to \mathcal{B}\mathsf{un}$ is closed-coCartesian.

□

Definition 2.42 ($\mathcal{B}\mathsf{un}_{\leq n}$)**.** We define the tangential structure $\mathcal{E}\mathsf{xit}_{\leq n} \to \mathcal{E}\mathsf{xit}$ as the full ∞-category of those $(x \in X)$ for which the local dimension $\mathsf{dim}_x(X) \leq n$ is bounded. We denote the resulting ∞-category

$$\mathcal{B}\mathsf{un}_{\leq n} ~:=~ \mathcal{B}\mathsf{un}^{\mathcal{E}\mathsf{xit}_{\leq n}} ~.$$

Remark 2.43. Explicitly, a functor $\mathsf{Exit}(K) \xrightarrow{(X \to K)} \mathcal{B}\mathsf{un}$ factors through $\mathcal{B}\mathsf{un}_{\leq n}$ if the fibers of $X \to K$ are bounded above in dimension by n. Indeed, the space of lifts of $\mathsf{Exit}(X) \to \mathcal{E}\mathsf{xit}$ through $\mathcal{E}\mathsf{xit}_{\leq n}$ is either empty or contractible, depending on the dimension of X.

Lemma 2.44. *For each dimension n, the functor $\mathcal{E}\mathsf{xit}_{\leq n} \to \mathcal{E}\mathsf{xit}$ is a tangential structure.*

Proof. Lemma 1.21 recalls from [AFR] that the restriction $\mathcal{E}\mathsf{xit}_{|\mathcal{B}\mathsf{un}^{\mathsf{cls}}} \to \mathcal{B}\mathsf{un}^{\mathsf{cls}}$ is the unstraightening construction (§3.2 of [Lu1]) applied to the functor

$$(\mathcal{B}\mathsf{un}^{\mathsf{cls}})^{\mathsf{op}} \underset{\text{Def } 1.18}{\simeq} \mathcal{S}\mathsf{trat}^{\mathsf{p.cbl.inj}} \xrightarrow{\mathsf{Exit}} \mathsf{Cat}_\infty \ .$$

So consider a proper constructible injection $X_0 \xrightarrow{c} X$. We must show, for each $x \in X_0$, that a bound in the local dimension $\mathsf{dim}_{c(x)}(X) \leq n$ implies a bound $\mathsf{dim}_x(X_0) \leq n$. This follows using that c is an embedding so that there is an inequality of local dimensions $\mathsf{dim}_x(X_0) \leq \mathsf{dim}_{c(x)}(X)$. □

Notation 2.45. For each tangential structure τ, and for each dimension n, we define the tangential structure $\tau_{\leq n} := \tau \underset{\mathcal{E}\mathsf{xit}}{\times} \mathcal{E}\mathsf{xit}_{\leq n}$ and use the notation

$$\mathcal{B}\mathsf{un}^{\tau}_{\leq n} := \mathcal{B}\mathsf{un}^{\tau_{\leq n}} \ .$$

Construction 2.46 ($\tau = \mathcal{S}$)**.** The assignment

$$\mathsf{Cat}_\infty \ni \mathcal{S} \ \mapsto \ \big(\mathcal{S} \times \mathcal{E}\mathsf{xit} \xrightarrow{\mathsf{pr}} \mathcal{E}\mathsf{xit}\big)$$

describes a functor from the ∞-category Cat_∞ of ∞-categories to the ∞-category of tangential structures. For $\mathcal{S}$ an ∞-category, we use the simplified notation

$$\mathcal{B}\mathsf{un}^{\mathcal{S}} \ := \ \mathcal{B}\mathsf{un}^{\mathcal{S}\times\mathcal{E}\mathsf{xit}} \ .$$

The next observation is direct from the construction $\tau \mapsto \mathcal{B}\mathsf{un}^{\tau}$.

Observation 2.47. Let $\mathcal{S}$ be an ∞-category. For each functor $\mathcal{K} \to \mathcal{B}\mathsf{un}$, the ∞-category of sections

$$\mathsf{Fun}_{/\mathcal{B}\mathsf{un}}\big(\mathcal{K}, \mathcal{B}\mathsf{un}^{\mathcal{S}}\big) \ \subset \ \mathsf{Fun}\big(\mathcal{E}\mathsf{xit}_{|\mathcal{K}}, \mathcal{S}\big)$$

is identified as the full ∞-subcategory consisting of those functors $\mathcal{F}\colon \mathcal{E}\mathsf{xit}_{|\mathcal{K}} \to \mathcal{S}$ for which the restriction $\mathcal{F}_{|\,\mathsf{cls}}\colon \mathcal{E}\mathsf{xit}^{\mathsf{cls}}_{|\mathcal{K}} \to \mathcal{S}$ factors through $\mathcal{S}^{\sim}$, the maximal ∞-subgroupoid of $\mathcal{S}$. In particular, an object of the ∞-category $\mathcal{B}\mathsf{un}^{\mathcal{S}}$ is a stratified space X together with a functor $\mathsf{Exit}(X) \xrightarrow{g} \mathcal{S}$.

2.5. **Vertical vari-framings.** Finally, we consider vari-framed stratified spaces, and variations thereon, as they organize as an ∞-category. These ∞-categories are designed so that they classify constructible families of vari-framed stratified spaces. In ordinary differential topology, a family of framed n-manifolds can be taken as a smooth fiber bundle $E \xrightarrow{\pi} B$ together with a trivialization of the vertical tangent bundle: $\mathsf{Ker}(\mathsf{T}_E \xrightarrow{D\pi} \pi^*\mathsf{T}_B) \underset{\varphi}{\simeq} \epsilon^n_E$. We imitate this definition simply by replacing the fiber bundle by a constructible bundle between stratified spaces.

We now introduce the primary ∞-categories of this article.

Definition 2.48 (Vari-framing)**.** The *vari-framing* tangential structure is the projection from the pullback

$$\begin{tikzcd}
\mathsf{vfr} \arrow[rr] \arrow[d] & & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \arrow[d, "\mathsf{diag}"] \\
\mathcal{E}\mathsf{xit} \arrow[r, "\mathsf{diag}"] & \mathcal{E}\mathsf{xit} \times \mathcal{E}\mathsf{xit} \arrow[r, "\epsilon^{\mathsf{dim}} \times \mathsf{T}"] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \times \mathcal{V}\mathsf{ect}^{\mathsf{inj}} .
\end{tikzcd}$$

For each dimension n, the *n-vari-framing* tangential structure is the pullback

$$\begin{tikzcd}
\mathsf{vfr}_n \arrow[r] \arrow[d] \arrow[dr] & \mathsf{vfr} \arrow[d] \\
\mathcal{E}\mathsf{xit}_{\leq n} \arrow[r] & \mathcal{E}\mathsf{xit} \ .
\end{tikzcd}$$

The ∞-category of *vari-framed (stratified) n-manifolds* is

$$\mathcal{M}\mathsf{fd}^{\mathsf{vfr}}_n \ := \ \mathcal{B}\mathsf{un}^{\mathsf{vfr}_n} \ .$$

It is immediate from definitions that the functor $\mathsf{vfr} \to \mathcal{E}\mathsf{xit}$, as well as $\mathsf{vfr}_n \to \mathcal{E}\mathsf{xit}$ for each $n \geq 0$, is indeed a tangential structure.

Remark 2.49. An object of $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ is a stratified space X of dimension bounded above by n together with an equivalence $\epsilon_X^{\mathsf{dim}} \underset{\varphi}{\simeq} \mathsf{T}_X$. In particular, the underlying topological space of X need not be a topological manifold, let alone of dimension n. Our choice for this terminology for the ∞-category $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ reflects the examples of objects therein that drive our interest: those (X, φ) for which X is a refinement of a smooth n-manifold.

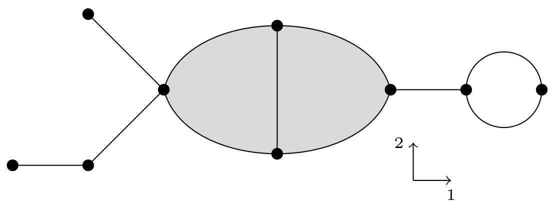

FIGURE 1. An object of $\mathcal{M}\mathsf{fd}_2^{\mathsf{sfr}}$.

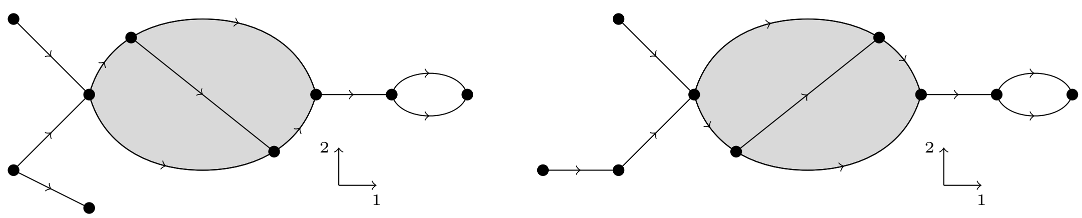

FIGURE 2. Two non-equivalent objects of $\mathcal{M}\mathsf{fd}_2^{\mathsf{vfr}}$ over the same object of $\mathcal{M}\mathsf{fd}_2^{\mathsf{sfr}}$ depicted in Figure 1.

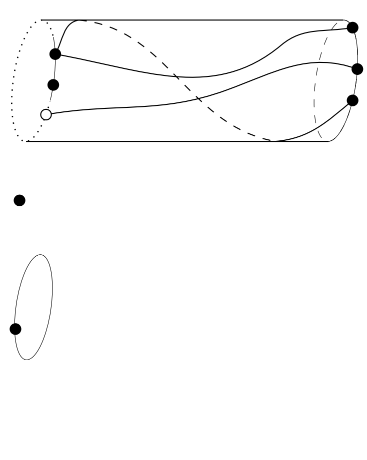

FIGURE 3. A morphism in $\mathcal{M}\mathsf{fd}_1$, depicted as a constructible bundle over Δ^1.

Example 2.50. For each $0 \leq i \leq n$, the hemispherical i-disk $\mathbb{D}^i$ of Example 2.20 is a vari-framed n-manifold. The boundary $\partial\mathbb{D}^i$ too is a vari-framed n-manifold. The inclusion of stratified spaces $c\colon \partial\mathbb{D}^i \hookrightarrow \mathbb{D}^i$ is an injective constructible bundle. The reversed mapping cylinder is constructible

bundle $\mathsf{Cylr}(c) \to \Delta^1$, thereby defining a morphism in $\mathsf{c}\mathcal{B}\mathsf{un}$ which is closed. The canonical functor $\mathsf{Exit}(\mathsf{Cylr}(c)) \to \mathcal{E}\mathsf{xit}$ is equipped with a lift to vfr, thereby defining a closed morphism in $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$:

$$c\colon \mathbb{D}^i \longrightarrow \partial\mathbb{D}^i ~.$$

Similarly, for each $n \geq i \geq j \geq 0$, the standard projection $a\colon \mathbb{D}^i \to \mathbb{D}^j$ between stratified spaces is a surjective constructible bundle. The canonical functor $\mathsf{Exit}(\mathsf{Cylr}(a)) \to \mathcal{E}\mathsf{xit}$ is equipped with a lift to vfr. This data defines a creation morphism in $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$:

$$a\colon \mathbb{D}^j \longrightarrow \mathbb{D}^i ~.$$

Remark 2.51. The ∞-category $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ captures several functorialities. We summarize these as the following monomorphisms (see §5.1 for a discussion of monomorphisms – it is an ∞-categorical notion of 'faithful representation').

- For each vari-framed stratified space X of dimension bounded above by n, there is a monomorphism
$$\mathsf{B}\,\mathsf{Aut}^{\mathsf{vfr}}(X) ~\hookrightarrow~ \mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}} ~.$$
In particular, for M a smooth framed n-manifold, there is a monomorphism $\mathsf{B}\,\mathsf{Diff}^{\mathsf{fr}}(M) \hookrightarrow \mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$. Therefore, a functor from $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ determines continuous representations of $\mathsf{Diff}^{\mathsf{fr}}(M)$ for each framed n-manifold M.
- For each vari-framed n-manifold M, there is a monomorphism from the moduli space of open embeddings
$$\Bigl\{U \underset{\mathsf{emb}}{\hookrightarrow} M\Bigr\} ~\hookrightarrow~ \mathcal{M}\mathsf{fd}_{n/M}^{\mathsf{vfr}} ~.$$
In this way, locality with respect to open covers can be accounted for.
- For each vari-framed n-manifold M, there is a monomorphism from the moduli space of constructible proper embeddings
$$\Bigl\{X \underset{\mathsf{p.cbl.emb}}{\hookrightarrow} M\Bigr\} ~\hookrightarrow~ (\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}})^{M/} ~.$$
In this way, locality with respect to cutting along strata can be accounted for.
- For each smooth framed k-manifold B, with $k \leq n$, is a monomorphism from the moduli space of smooth framed proper codimension-$(n-k)$ fiber bundles
$$\Bigl\{E^n \xrightarrow[\mathsf{p.sm.bdl}]{\pi} B ~,~ \mathsf{Ker}(D\pi) \underset{\varphi}{\simeq} \epsilon_E^{n-k}\Bigr\} ~\hookrightarrow~ (\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}})^{B/} ~.$$
Therefore, a functor from $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ determines transfer-type maps for each such framed fiber bundle.
- For each smooth framed n-manifold $M = (M,\varphi)$, there is a monomorphism from the moduli space of vari-framed refinements
$$\Bigl\{\widetilde{M} \xrightarrow[\mathsf{ref}]{} M ~,~ \mathsf{T}_{\widetilde{M}} \underset{\widetilde{\varphi}/\varphi}{\simeq} \epsilon_{\widetilde{M}}^{\mathsf{dim}}\Bigr\} ~\hookrightarrow~ \mathcal{M}\mathsf{fd}_{n/M}^{\mathsf{vfr}} ~.$$
Therefore, a functor from $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ determines composition-type maps for each vari-framed refinement.

Observation 2.52. Since the construction $\tau \mapsto \mathcal{B}\mathsf{un}^{\tau}$ is a right adjoint, there are canonical pullback diagrams among ∞-categories

$$\begin{tikzcd}
\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}} \arrow[r] \arrow[d] & \mathcal{B}\mathsf{un}^{\mathsf{vfr}} \arrow[r] \arrow[d] & \mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}} \arrow[d] \\
\mathcal{B}\mathsf{un}_{\leq n} \arrow[r] & \mathcal{B}\mathsf{un} \arrow[r, "(\epsilon^{\mathsf{vdim}}{,}\mathsf{T}^{\mathsf{v}})"] & \mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}} \times \mathcal{V}\mathsf{ect}^{\mathsf{inj}}}
\end{tikzcd} ~.$$

We now introduce ∞-categories of stratified spaces equipped with *stable* tangential structures, as they vary in constructible families.

Definition 2.53 (B-framings)**.** For each dimension n, and for each map of spaces $B \to \mathsf{BO}(n)$, the *B-framing* tangential structure is the limit

$$\begin{tikzcd}
\mathsf{s}B \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[d] \\
& \mathsf{Ar}(\mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \arrow[r, "\mathsf{ev}_t"] \arrow[d, "\mathsf{ev}_s"] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \\
\mathcal{E}\mathsf{xit} \arrow[r, "\mathsf{T}"] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} &
\end{tikzcd}$$

as it is equipped with the left vertical projection, where the right vertical map is the composition $B \to \mathsf{BO}(n) \hookrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$. The ∞-category of *B-framed (stratified) n-manifolds* is

$$\mathcal{M}\mathsf{fd}_n^B \ := \ \mathcal{B}\mathsf{un}^{\mathsf{s}B} \ .$$

The ∞-category of *(stratified) n-manifolds*, and the ∞-category of *solidly framed (stratified) n-manifolds*, are the special cases

$$\mathcal{M}\mathsf{fd}_n \ := \ \mathcal{M}\mathsf{fd}_n^{\mathsf{BO}(n)} \qquad \text{and} \qquad \mathcal{M}\mathsf{fd}_n^{\mathsf{sfr}} \ := \ \mathcal{M}\mathsf{fd}_n^{*} \ .$$

It is immediate from definitions that each functor $\mathsf{s}B \to \mathcal{E}\mathsf{xit}$ is indeed a tangential structure. Note that the assignment

$$\mathcal{S}\mathsf{paces}_{/\,\mathsf{BO}(n)} \ni \big(B \to \mathsf{BO}(n)\big) \ \mapsto \ \mathcal{M}\mathsf{fd}_n^B \in \mathsf{Cat}_{\infty/\,\mathcal{B}\mathsf{un}}$$

is manifestly functorial. Also notice, for n a dimension, the pullback diagram among ∞-categories

$$\begin{tikzcd}
\mathsf{sfr}_n \arrow[rr] \arrow[d] & & \mathsf{Ar}(\mathcal{V}\mathsf{ect}^{\mathsf{inj}}) \arrow[d, "(\mathsf{ev}_s,\mathsf{ev}_t)"] \\
\mathcal{E}\mathsf{xit} \arrow[r, "\mathsf{diag}"] & \mathcal{E}\mathsf{xit} \times \mathcal{E}\mathsf{xit} \arrow[r, "\mathsf{T}\times\epsilon^n"] & \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \times \mathcal{V}\mathsf{ect}^{\mathsf{inj}}
\end{tikzcd}$$

in which ϵ^n is the constant functor at $\mathbb{R}^n$. Because $n \in [n]$ is terminal, there is a canonical morphism of functors $\mathcal{E}\mathsf{xit}_{|\leq n} \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ from the restriction of the dimension bundle

$$\epsilon^{\mathsf{dim}}{}_{|\,\mathcal{E}\mathsf{xit}_{\leq n}} \longrightarrow \epsilon^n$$

This yields the following.

Observation 2.54. For each dimension n, there are projections

$$\mathcal{B}\mathsf{un}^{\mathsf{vfr}} \ \hookleftarrow \ \mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}} \ \to \ \mathcal{M}\mathsf{fd}_n^{\mathsf{sfr}} \ .$$

3. Disks

A stratified space naturally accommodates two types of gluing procedures: unions of open subsets, thereby making use of the underlying topology; splicing along strata, thereby making use of the stratification. Correspondingly, there are two notions of locality for invariants of stratified spaces. These are interrelated. For instance, should a stratified space be sufficiently finely stratified, then these two localities refine each other, in a locally constant sense: regular neighborhoods of strata determines an open cover which, up to isotopy, refines any other open cover. We articulate this intuition of 'sufficiently finely stratified' as a *disk*-stratification. To define the notion of a disk-stratification we consider suspensions of compact stratified spaces, and suspensions of structures thereon. To do so efficiently, we introduce such suspension by way of *wreath* product.

3.1. **Iterated constructible bundles.** Here we exploit the universal nature of $\mathcal{B}\mathsf{un}$ as a classifying object for constructible bundles. To this end, we utilize the point-set entity Bun from which $\mathcal{B}\mathsf{un}$ is derived (see [AFR], or §6.3 for a review). The main output of this section is Definition 3.5 which establishes the functor $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$ that we will make ongoing use of.

Consider the simplicial category

$$\mathsf{Bun}^\bullet \colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$$

whose category of p-simplices is the subcategory of $\mathsf{Fun}([p]^{\mathsf{op}}, \mathsf{Strat})$ whose objects are those $X_\bullet \colon [p]^{\mathsf{op}} \to \mathsf{Strat}$ for which, for each $0 \leq i \leq j \leq p$, $X_j \to X_i$ is a constructible bundle; the morphisms $X_\bullet \to Y_\bullet$ are those natural transformations for which, for each $0 \leq i \leq j \leq p$, the square

$$\begin{tikzcd} X_j \arrow[r] \arrow[d] & Y_j \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd}$$

is a pullback. The simplicial structure functors of $\mathsf{Fun}([\bullet]^{\mathsf{op}}, \mathsf{Strat})$ restrict to $\mathsf{Bun}^\bullet$, using the result from §6 of [AFR] that constructible bundles compose.

Among the simplicial structure functors, restriction along each $\{0 < \cdots < i\} \hookrightarrow \{0 < \cdots < p\}$ induces a right fibration

$$\mathsf{Bun}^p \longrightarrow \mathsf{Bun}^i ~. \tag{9}$$

Observation 3.1. The projection (9) for the case $i = 0$ is a transversality sheaf. This follows by induction on p, after the base case $p = 1$ which is proved in §6 of [AFR].

Notation 3.2 ($\mathcal{B}\mathsf{un}^p$)**.** After Observation 3.1, the main result of [AFR] associates to each such functor $\mathsf{Bun}^p \to \mathsf{Bun}^0 = \mathsf{Strat}$ an ∞-category

$$\mathcal{B}\mathsf{un}^p ~.$$

Heuristically, the ∞-category $\mathcal{B}\mathsf{un}^p$ classifies p-fold sequences of constructible bundles.

Observation 3.3. Consider the subcategory $\mathbf{\Delta}_- \subset \mathbf{\Delta}$ consisting of the same objects and morphisms that preserve minimima. The ∞-categories $\mathcal{B}\mathsf{un}^p$ assemble as a functor

$$\mathcal{B}\mathsf{un}^\bullet \colon \mathbf{\Delta}_-^{\mathsf{op}} \to \mathsf{Cat}_\infty ~.$$

Observation 3.4. Consider a functor $\mathsf{Exit}(K) \xrightarrow{(Y \to K)} \mathcal{B}\mathsf{un}$ classifying the indicated constructible bundle. From the Construction 2.46 of $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}}$, there is a canonical monomorphism spaces of functors

$$\mathsf{Map}_{/\mathcal{B}\mathsf{un}}\bigl(\mathsf{Exit}(K), \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}}\bigr) ~\subset~ \mathsf{Map}\bigl(\mathsf{Exit}(Y), \mathcal{B}\mathsf{un}\bigr)$$

which is contravariantly functorial in the variable K. As so, we have the following description of the space of functors

$$\mathsf{Map}\bigl(\mathsf{Exit}(K), \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}}\bigr) ~\simeq~ |\{X \xrightarrow{\mathsf{cbl}} Y \xrightarrow{\mathsf{cbl}} K \times \Delta^\bullet_e\}| ~.$$

Better, for each $p > 0$, there is a canonical equivalence of ∞-categories

$$\mathcal{B}\mathsf{un}^p ~\simeq~ \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{p-1}} ~.$$

Observations 3.3 and 3.4 combine, which we highlight as the following.

Definition 3.5. For each $n > 0$, the functor between ∞-categories

$$\circ \colon ~\mathcal{B}\mathsf{un}^n \longrightarrow \mathcal{B}\mathsf{un}$$

is the assignment of K-points

$$\bigl(\mathsf{Exit}(K) \xrightarrow{(X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_1} K)} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}}\bigr) \mapsto \bigl(\mathsf{Exit}(K) \xrightarrow{(X_n \xrightarrow{\pi_1 \circ \cdots \circ \pi_n} K)} \mathcal{B}\mathsf{un}\bigr) ~.$$

Because each distinguished class ψ of morphisms in $\mathcal{B}\mathsf{un}$ is closed under composition, the restriction of this functor factors:

$$\mathcal{B}\mathsf{un}^{\psi,\mathcal{B}\mathsf{un}^{\psi}} \longrightarrow \mathcal{B}\mathsf{un}^{\psi} \ .$$

Finally, we note that the results in this seciton are equally valid upon replacing the role of constructible bundles by *proper* constructible bundles. We highlight this as the following.

Observation 3.6. There is an ∞-subcategory $\mathsf{c}\mathcal{B}\mathsf{un}^n$ of $\mathcal{B}\mathsf{un}^n$ that classifies n-fold sequences of *proper* constructible bundles. Furthermore, there is a factorization

$$\mathsf{c}\mathcal{B}\mathsf{un}^n \xrightarrow{\circ} \mathsf{c}\mathcal{B}\mathsf{un}$$

of the restriction of $\mathcal{B}\mathsf{un}^n \xrightarrow{\circ} \mathcal{B}\mathsf{un}$.

3.2. **Iterated framings.** Here we show that the functor $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$ of Definition 3.5 respects various notions of framings. This is articulated as Corollary 3.11 which establishes a functor $\mathcal{B}\mathsf{un}^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathsf{vfr}}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}^{\mathsf{vfr}}$. In brief, this is a functorial construction of a vertical vari-framing on a constructible bundle $X \to K$ for each factorization through vertically vari-framed constructible bundles $X \to Y \to K$.

We first show that the parametrizing ∞-category $\mathcal{E}\mathsf{xit}$ respects the functor $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$ of Definition 3.5.

Lemma 3.7. *There is a filler in the diagram among ∞-categories*

$$\begin{tikzcd}
\mathcal{E}\mathsf{xit} \arrow[d] & \mathcal{E}\mathsf{xit}_{|\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}}} \arrow[l, dashed, "\mathsf{pr}"'] \arrow[r] \arrow[d] & \mathcal{E}\mathsf{xit} \arrow[d] \\
\mathcal{B}\mathsf{un} & \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \arrow[l, "\mathsf{pr}"'] \arrow[r, "\circ"] & \mathcal{B}\mathsf{un}\, .
\end{tikzcd}$$

Proof. As striation sheaves, this filler is the map of presheaf on Strat for which, for each stratified space K, the map on spaces of K-points is the assignment

$$\begin{tikzcd}
X \times \Delta^{\bullet}_{e} \arrow[r, "q"'] & Y \times \Delta^{\bullet}_{e} \arrow[r, "p"'] & K \times \Delta^{\bullet}_{e} \arrow[ll, bend right, "\sigma"']
\end{tikzcd}
\qquad \mapsto \qquad
\begin{tikzcd}
X \times \Delta^{\bullet}_{e} \arrow[r, "pq"'] & K \times \Delta^{\bullet}_{e} \arrow[l, bend right, "\sigma"']
\end{tikzcd} .$$

□

The next result articulates how, for $\mathcal{A}$ and $\mathcal{B}$ ∞-categories, $\mathcal{A}$-structures combine with $\mathcal{B}$-structures over the functor $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$ of Definition 3.5.

Corollary 3.8. *Each functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ among ∞-categories canonically determines a filler in the diagram among ∞-categories*

$$\begin{tikzcd}
\mathcal{B}\mathsf{un}^{\mathcal{A}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathcal{B}}} \arrow[r, dashed] \arrow[d] & \mathcal{B}\mathsf{un}^{\mathcal{C}} \arrow[d] \\
\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \arrow[r, "\circ"] & \mathcal{B}\mathsf{un}\, .
\end{tikzcd}$$

Proof. Through Observation 2.47, which explicates what $\mathcal{B}\mathsf{un}^{\mathcal{S}}$ classifies for each ∞-category $\mathcal{S}$, the problem is to canonically construct a functor

$$\mathcal{E}\mathsf{xit}_{|\mathcal{B}\mathsf{un}^{\mathcal{A}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathcal{B}}}} \longrightarrow \mathcal{C}$$

satisfying the locality of Observation 2.47. This locality will be manifest from the construction of the functor. Lemma 3.7, just above, offers the functor

$$\mathcal{E}\mathsf{xit}_{|\mathcal{B}\mathsf{un}^{\mathcal{A}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathcal{B}}}} \longrightarrow \mathcal{E}\mathsf{xit}_{|\mathcal{B}\mathsf{un}^{\mathcal{A}}} \times \mathcal{E}\mathsf{xit}_{|\mathcal{B}\mathsf{un}^{\mathcal{B}}} \ .$$

The counit of the adjunction defining $\tau \mapsto \mathcal{B}\mathsf{un}^{\tau}$ gives the functor
$$\mathcal{E}\mathsf{xit}_{|\,\mathcal{B}\mathsf{un}^{\mathcal{A}}} \times \mathcal{E}\mathsf{xit}_{|\,\mathcal{B}\mathsf{un}^{\mathcal{B}}} \longrightarrow \mathcal{A} \times \mathcal{B}~.$$
The result follows by composing with $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$.

$\square$

Corollary 3.8 applied to the direct sum functor $\mathcal{V}\mathsf{ect}^{\mathsf{inj}} \times \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \xrightarrow{\oplus} \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ gives the next result.

Corollary 3.9. *Direct sum of vector spaces $\mathcal{V}\mathsf{ect}^{\mathsf{inj}} \times \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \xrightarrow{\oplus} \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ determines a functor*
$$\mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}}} \xrightarrow{~\circ~} \mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}}$$
over $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$.

The next result articulates how the functor of Corollary 3.9 just above respects the vertical constructible tangent bundle as well as the vertical dimension constructible bundle.

Lemma 3.10. *The diagram among ∞-categories*
$$\begin{tikzcd}
\mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}}} \arrow[d, "\circ"'] & \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \arrow[l, "(\epsilon^{\mathsf{vdim}}, \mathcal{B}\mathsf{un}^{\epsilon^{\mathsf{vdim}}})"'] \arrow[r, "(\mathsf{T}^{\mathsf{v}}, \mathcal{B}\mathsf{un}^{\mathsf{T}^{\mathsf{v}}})"] \arrow[d, "\circ"'] & \mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}}} \arrow[d, "\circ"] \\
\mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}} & \mathcal{B}\mathsf{un} \arrow[l, "\epsilon^{\mathsf{vdim}}"'] \arrow[r, "\mathsf{T}^{\mathsf{v}}"] & \mathcal{B}\mathsf{un}^{\mathcal{V}\mathsf{ect}^{\mathsf{inj}}}
\end{tikzcd}$$
commutes.

Proof. In this proof we will use the following notation.

For $X \xrightarrow{p} K$ a constructible bundle, we use the notations
$$\mathsf{T}_p \colon \mathsf{Exit}(X) \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}} \qquad \text{and} \qquad \epsilon_p^{\mathsf{dim}_{(-)}(p^{-1}(p(-))}\colon \mathsf{Exit}(X) \longrightarrow \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$$
for the restrictions of T^{v} and ϵ^{vdim} along $\mathsf{Exit}(X) \simeq \mathcal{E}\mathsf{xit}_{|\,\mathsf{Exit}(K)} \hookrightarrow \mathcal{E}\mathsf{xit}$.

Fix a K-point $X \xrightarrow{q} Y \xrightarrow{p} K$ of Bun^2. Inspecting their definitions, there are short exact sequences of $\mathcal{V}\mathsf{ect}$-valued functors from $\mathsf{Exit}(X)$
$$0 \longrightarrow \mathsf{T}_q \longrightarrow \mathsf{T}_{pq} \xrightarrow{Dq} q^*\mathsf{T}_q \longrightarrow 0$$
and
$$0 \longrightarrow \epsilon_X^{\mathsf{dim}_{(-)}(q^{-1}q(-))} \longrightarrow \epsilon_X^{\mathsf{dim}_{(-)}(pq^{-1}(pq(-))} \longrightarrow q^*\epsilon_Y^{\mathsf{dim}_{(-)}(p^{-1}(p(-))} \longrightarrow 0~.$$
Because each of these exact sequences is comprised functors valued in *finite dimensional* vector spaces, they each canonically split in $\mathcal{V}\mathsf{ect}$. This is to say there are canonical identifications in $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$:
$$\mathsf{T}_q \oplus q^*\mathsf{T}_q ~\simeq~ \mathsf{T}_{pq} \qquad \text{and} \qquad \epsilon_X^{\mathsf{dim}_{(-)}(q^{-1}q(-))} \oplus q^*\epsilon_Y^{\mathsf{dim}_{(-)}(p^{-1}(p(-))} ~\simeq~ \epsilon_X^{\mathsf{dim}_{(-)}(pq^{-1}(pq(-))}~.$$
This verifies the asserted commutativity for each K-point of Bun^2. The asserted commutavity follows because the aformentioned short exact sequences manifestly pullback along morphisms $K \to K'$.

$\square$

The next result is the culmination of the formal manipulations above. It articulates a sense in which the functor $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$ respects certain tangential structures of particular interest.

Corollary 3.11. *Let n and k be dimensions. Let $B \to \mathsf{BO}(n)$ and $B' \to \mathsf{BO}(k)$ and $B'' \to \mathsf{BO}(n+k)$ be maps of spaces, equipped with a map $B \times B' \to B''$ over $\mathsf{BO}(n) \times \mathsf{BO}(k) \xrightarrow{\oplus} \mathsf{BO}(n+k)$. There are preferred functors over $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$:*
$$\mathcal{B}\mathsf{un}^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}^{\mathsf{vfr}}} \xrightarrow{~\circ~} \mathcal{B}\mathsf{un}^{\mathsf{vfr}} \qquad \textit{and} \qquad \mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{M}\mathsf{fd}_k^{\mathsf{vfr}}} \xrightarrow{~\circ~} \mathcal{M}\mathsf{fd}_{n+k}^{\mathsf{vfr}}$$

as well as

$$\mathcal{M}\mathsf{fd}_n^B \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{M}\mathsf{fd}_k^{B'}} \xrightarrow{\circ} \mathcal{M}\mathsf{fd}_{n+k}^{B''} .$$

Proof. The first and second functors are constructed directly by unwinding definitions, making use of Observation 2.52 which gives $\mathcal{B}\mathsf{un}^{\mathsf{vfr}}$, and thereafter $\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$, as a certain pullback. The third functor is constructed similarly.

□

3.3. **Suspension.** We construct a suspension of framed stratified spaces. This formation will play an important role in our definition of disk-stratifications, for instance, by constructing cells in a way amenable to inductive on dimension. The main result in this section is Lemma 3.17, which states that every automorphism of a suspended vari-framed stratified space L is the suspension of an automorphism of L. This result reflects the rigidity of vari-framings, which ultimately is responsible for the contact between higher categories and vari-framed differential topology.

For the next definition we make referene to the vari-framed stratified space $\mathbb{D}^1$ of Example 2.20. The underlying space of $\mathbb{D}^1$ is $[-1,1]$, and as so it is equipped with maps $\{-1\} \to \mathbb{D}^1 \leftarrow \{1\}$, each of which is a constructible proper embedding.

Definition 3.12. The *suspension* of a compact stratified space X is the iterated pushout in Strat:

$$\mathsf{S}(X) \ := \ * \coprod_{X\times\{-1\}} X \times \mathbb{D}^1 \coprod_{X\times\{1\}} * \ .$$

The *fiberwise suspension* of a proper constructible bundle $X \to K$ is the iterated pushout in Strat:

$$\mathsf{S}^{\mathsf{fib}}(X) \ := \ K \coprod_{X\times\{-1\}} X \times \mathbb{D}^1 \coprod_{X\times\{1\}} K \ .$$

Observation 3.13. For each proper constructible bundle $X \to K$, there is a natural factorization of the projection

$$\mathsf{S}^{\mathsf{fib}}(X) \longrightarrow K \times \mathbb{D}^1 \xrightarrow{\mathsf{pr}} K$$

through proper constructible bundles. Furthermore, for each conically smooth map $K' \to K$, the diagram of stratified spaces

$$\begin{tikzcd}
\mathsf{S}^{\mathsf{fib}}(X_{|K'}) \arrow[r] \arrow[d] & K' \times \mathbb{D}^1 \arrow[r] \arrow[d] & K' \arrow[d] \\
\mathsf{S}^{\mathsf{fib}}(X) \arrow[r] & K \times \mathbb{D}^1 \arrow[r] & K
\end{tikzcd}$$

is comprised of pullback squares.

The next result articulates a sense in which suspension repects vari-framings as well as solid framings.

Lemma 3.14. *The assignment*

$$(X \underset{\mathsf{p.cbl}}{\longrightarrow} K) \mapsto \Big(\mathsf{S}^{\mathsf{fib}}(X) \longrightarrow K \times \mathbb{D}^1 \longrightarrow K\Big) \tag{10}$$

defines a functor

$$\mathsf{S}\colon \mathsf{c}\mathcal{B}\mathsf{un} \longrightarrow \mathsf{c}\mathcal{M}\mathsf{fld}_1^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}} \ .$$

This functor admits lifts

$$\mathsf{S}^{\mathsf{fr}}\colon \mathsf{c}\mathcal{B}\mathsf{un}^{\mathsf{vfr}} \longrightarrow \mathsf{c}\mathcal{M}\mathsf{fld}_1^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}^{\mathsf{vfr}}} \qquad \text{and} \qquad \mathsf{S}^{\mathsf{fr}}\colon \mathsf{c}\mathcal{B}\mathsf{un}^{\mathsf{sfr}_n} \longrightarrow \mathsf{c}\mathcal{M}\mathsf{fld}_1^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}^{\mathsf{sfr}_{n+1}}}$$

for each dimension n.

Proof. Observation 3.13 gives that the assignment (10) defines a functor $\mathsf{cBun} \to \mathsf{Bun}^2$ over Strat, which takes values in composable *proper* constructible bundles. Applying the topologizing diagram, there results a functor $\mathsf{c}\mathcal{B}\mathsf{un} \to \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}}$ between ∞-categories, regarded here as striation sheaves through the main result of [AFR]. The composite functor $\mathsf{c}\mathcal{B}\mathsf{un} \to \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}} \xrightarrow{\mathsf{pr}} \mathcal{B}\mathsf{un}$ is constant at $\mathbb{D}^1$. The preferred vari-framing of $\mathbb{D}^1$ determines a lift

$$\mathsf{c}\mathcal{B}\mathsf{un} \longrightarrow \mathsf{c}\mathcal{M}\mathsf{fld}_1^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}}$$

as desired.

Now, consider a proper constructible bundle $X \xrightarrow{\pi} K$. We must explain how each vertical vari-framing of $X \to K$ canonically determines a vertical vari-framing of $\mathsf{S}^{\mathsf{fib}}(X) \to K \times \mathbb{D}^1$. It will be clear that the vertical vari-framing pulls back among stratified maps of the K argument. Restriction along the equator of the fiberwise suspension determines the map between spaces of lifts over $\mathcal{E}\mathsf{xit}$

$$\mathsf{vfr}\big(\mathsf{S}^{\mathsf{fib}}(X) \to K \times \mathbb{D}^1\big) \longrightarrow \mathsf{vfr}(X \to K)~. \tag{11}$$

We will explain that this map is an equivalence. Using the results of §3 of [AFR], the double pushout defining the fiberwise suspension is preserved by the exit-path functor:

$$\mathsf{Exit}(K) \underset{\mathsf{Exit}(X)\times\{0\}}{\amalg} \mathsf{Exit}(X) \times [1] \underset{\mathsf{Exit}(X)\times\{1\}}{\amalg} \mathsf{Exit}(K) \xrightarrow{\simeq} \mathsf{Exit}\big(\mathsf{S}^{\mathsf{fib}}(X)\big)$$

over $\mathsf{Exit}(K) \times [1] \xrightarrow{\simeq} \mathsf{Exit}(K \times \mathbb{D}^1)$. Because the fibers of $\mathsf{S}^{\mathsf{fib}}(X) \to K \times \mathbb{D}^1$ over $K \times \partial\mathbb{D}^1$ are terminal, both T and ϵ^{dim} restrict along $\mathsf{Exit}(K) \times \partial[1] \hookrightarrow \mathsf{Exit}\big(\mathsf{S}^{\mathsf{fib}}(X)\big)$ as the constant functor to $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ valued at the zero vector space. Because the zero vector space is initial in $\mathcal{V}\mathsf{ect}^{\mathsf{inj}}$, the restriction map (11) is an equivalence of spaces.

□

In ordinary differential topology, an orientation on M determines an orientation on $M \times \mathbb{R}$ in a standard manner. This is likewise the case for spin structures, as well as framings. In the abstract, these are the *data* of maps of spaces $B_n \to B_{n+1}$ over the standard map $\mathsf{BO}(n) \xrightarrow{-\times\mathbb{R}} \mathsf{BO}(n+1)$ for the cases $B_n = \mathsf{BSO}(n)$ and $B_n = \mathsf{BSpin}(n)$ and $B_n = \mathsf{EO}(n)$. The next definition imitates such data for the general case of τ-structures on stratified spaces, so as to vary in constructible families.

Definition 3.15. A *suspending tangential structure* is a tangential structure τ together with a *framed suspension* functor

$$\mathsf{S}^{\mathsf{fr}}\colon \mathsf{c}\mathcal{B}\mathsf{un}^{\tau} \longrightarrow \mathcal{B}\mathsf{un}^{\tau}$$

over the composite functor $\mathsf{c}\mathcal{B}\mathsf{un} \xrightarrow{\mathsf{S}} \{\mathbb{D}^1\} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$.

Example 3.16. The functor of Lemma 3.14, given in terms of fiberwise suspensions, composed with the functor of Corollary 3.11, which articulates compatibilities with vari-framings, gives a framed suspension functor for the tangential structure vfr.

The next result captures the essential feature of vari-framings.

Lemma 3.17. *For each compact vari-framed stratified space L, the map between spaces of automorphisms*

$$\mathsf{S}^{\mathsf{fr}}\colon \mathsf{Aut}_{\mathcal{B}\mathsf{un}^{\mathsf{vfr}}}(L) \xrightarrow{\simeq} \mathsf{Aut}_{\mathcal{B}\mathsf{un}^{\mathsf{vfr}}}\big(\mathsf{S}^{\mathsf{fr}}(L)\big)$$

is an equivalence.

Proof. The contending equivalence is between the maximal connected ∞-subgroupoids of $\mathcal{B}\mathsf{un}^{\mathsf{vfr}}$ that contain the respective objects L and $\mathsf{S}^{\mathsf{fr}}(L)$. The maximal ∞-subgroupoid of $\mathcal{B}\mathsf{un}^{\mathsf{vfr}}$ corresponds, via the main result of [AFR], to the striation sheaf that classifies fiber bundles among stratified spaces which are equipped with a vertical vari-framing. Therefore, to prove the equivalence of the lemma it is enough to prove, for each smooth manifold T, that each T-point of the codomain lifts to an T-point of the domain. So consider an T-point of the striation sheaf $\mathsf{B}\,\mathsf{Aut}_{\mathcal{B}\mathsf{un}^{\mathsf{vfr}}}\big(\mathsf{S}^{\mathsf{fr}}(L)\big)$; it classifies a proper fiber bundle $E \xrightarrow{p} T$, equipped with a vertical vari-framing $\mathsf{Exit}(E) \xrightarrow{g} \mathsf{vfr}$, with

each structured fiber $\big(E_t, g_{|\,\mathsf{Exit}(E_t)}\big)$ is equivalent in $\mathcal{B}\mathsf{un}^{\mathsf{vfr}}$ to $\mathsf{S}^{\mathsf{fr}}(L)$. To construct the desired lift of this T-point to $\mathsf{B}\,\mathsf{Aut}_{\mathcal{B}\mathsf{un}^{\mathsf{vfr}}}(L)$ is the problem of constructing a vertically vari-framed proper fiber bundle $E_0 \to T$ and an equivalence $\mathsf{S}^{\mathsf{fib},\mathsf{fr}}(E_0) \simeq E$ of vertically vari-framed fiber bundles over T from the fiberwise framed suspension.

The fiberwise 0-dimension strata of E is a two-sheeted covering over T. The vertical vari-framing of $E \to T$ in particular implies this two-sheeted cover is trivial. We denote it as $T_- \sqcup T_+ \subset E$, with the subscripts marking if first coordinate agrees or disagrees with a collaring-coordinate about each cofactor. Taking the unzip along this closed constructible subspace gives the composable pair of proper constructible bundles

$$\mathsf{Unzip}_{T_-\sqcup T_+}(E) \xrightarrow{q} E \xrightarrow{p} T~.$$

Because T is a smooth manifold the functor $\mathsf{T}_T\colon \mathsf{Exit}(T) \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ factors through $\mathcal{V}\mathsf{ect}^{\sim}$, the maximal ∞-subgroupoid. Using that the constructible tangent bundle is a morphism $\mathsf{T}\colon \mathcal{E}\mathsf{xit} \to \mathcal{V}\mathsf{ect}^{\mathsf{inj}}$ of $\mathcal{V}\mathsf{ect}^{\sim}$-modules, there results the short exact sequence of $\mathcal{V}\mathsf{ect}$-valued functors from $\mathsf{Exit}\big(\mathsf{Unzip}_{T_-\sqcup T_+}(E)\big)$

$$0 \longrightarrow \mathsf{T}_q \longrightarrow \mathsf{T}_{pq} \longrightarrow q^*\mathsf{T}_p \longrightarrow 0~.$$

(Here, and through this proof, we use the notation established at the beginning of the proof of Lemma 3.10.) Restricted to $\mathsf{Exit}\big(\mathsf{Link}_{T_-\sqcup T_+}(E)\big)$, the cokernel term vanishes; restricted to

$$\mathsf{Exit}\big(\mathsf{Unzip}_{T_-\sqcup T_+}(E) \smallsetminus \mathsf{Link}_{T_-\sqcup T_+}(E)\big) \xrightarrow{\cong} \mathsf{Exit}(E \smallsetminus T_- \sqcup T_+)~,$$

the kernel term vanishes and the cokernel term does not. The first coordinate of the vertical vari-framing $\mathsf{Exit}(E) \to \mathsf{vfr}$ therefore determines a non-vanishing parallel vector field on $\mathsf{Unzip}_{T_-\sqcup T_+}(E) \smallsetminus \mathsf{Link}_{T_-\sqcup T_+}(E)$ in the sense of §8 of [AFT]. We will now extend this vector field to all of $\mathsf{Unzip}_{T_-\sqcup T_+}(E)$.

Let $\overline{\alpha}\colon \mathsf{Link}_{T_-\sqcup T_+}(E) \times [0,1) \hookrightarrow \mathsf{Unzip}_{T_-\sqcup T_+}(E)$ be a choice of collaring, the existence of which is guaranteed by the results in §8 of [AFT]. Denote the restriction $\alpha\colon \mathsf{Link}_{T_-\sqcup T_+}(E) \times \{\frac{1}{2}\} \to \mathsf{Unzip}_{T_-\sqcup T_+}(E) \smallsetminus \mathsf{Link}_{T_-\sqcup T_+}(E) \cong E \smallsetminus (T_- \sqcup T_+)$. The vertical vari-framing of $E \to T$ determines an identification of short exact sequences of $\mathcal{V}\mathsf{ect}$-valued functors from $\mathsf{Exit}\big(\mathsf{Link}_{T_-\sqcup T_+}(E)\big)$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, leftrightarrow] & \mathsf{T}^{\mathsf{v}}_{\mathsf{Link}_{T_-\sqcup T_+}(E)} \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & \alpha^*\mathsf{T}_{E\smallsetminus T_-\sqcup T_+} \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & \epsilon^1_{\mathsf{Link}_{T_-\sqcup T_+}(E)} \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & 0 \arrow[d, leftrightarrow] \\
0 \arrow[r] & \alpha^*\epsilon^{\mathsf{vdim}-1}_{E\smallsetminus T_-\sqcup T_+} \arrow[r] & \alpha^*\epsilon^{\mathsf{vdim}}_{E\smallsetminus T_-\sqcup T_+} \arrow[r] & \alpha^*\epsilon^1_{E\smallsetminus T_-\sqcup T_+} \arrow[r] & 0
\end{tikzcd}$$

The above diagram grants that this vector field extends to a non-vanishing vector field on $\mathsf{Unzip}_{T_-\sqcup T_+}(E)$ in a neighborhood of $\mathsf{Link}_{T_-\sqcup T_+}(E)$ which agrees with the one on $\mathsf{Unzip}_{T_-\sqcup T_+}(E) \smallsetminus \mathsf{Link}_{T_-\sqcup T_+}(E)$ constructed in the paragraph above.

Flowing by the vector field on $\mathsf{Unzip}_{T_-\sqcup T_+}(E)$ just constructed gives a partially defined conically smooth map

$$\gamma\colon~\mathsf{Unzip}_{T_-\sqcup T_+}(E) \times \mathbb{D}^1 \dashrightarrow \mathsf{Unzip}_{T_-\sqcup T_+}(E) \times \mathbb{D}^1$$

over $T \times \mathbb{D}^1$ which, upon applying $\mathsf{Exit}(-)$, lies over vfr. This map γ extends the map α above, and has the property that its restriction

$$\gamma_|\colon~\mathsf{Link}_{T_-}(E) \times \mathbb{D}^1 \longrightarrow \mathsf{Unzip}_{T_-\sqcup T_+}(E)$$

is defined and is an isomorphism over T and, upon applying $\mathsf{Exit}(-)$, it lies over vfr. In particular, we recognize an isomorphism of stratified spaces $\mathsf{S}^{\mathsf{fib},\mathsf{fr}}\big(\mathsf{Link}_{T_-}(E)\big) \cong E$ over T which, upon applying $\mathsf{Exit}(-)$, lies over vfr. The result follows from from the identification of vertically vari-framed constructible bundles $\mathsf{Link}_{T_-}(E) \simeq L$ over T.

□

The following definition is supported by Lemma 3.14, which articulates that suspensions naturally carry compact vari-framed stratified spaces to vari-framed stratified spaces.

Definition 3.18 (Hemispherical disks)**.** The *hemispherical n-disk* is the vari-framed n-manifold defined inductively as the framed suspension

$$\mathbb{D}^n \ := \ \mathsf{S}^{\mathsf{fr}}(\mathbb{D}^{n-1}) \ \in \ \mathsf{c}\mathcal{B}\mathsf{un}^{\mathsf{vfr}}$$

with $\mathbb{D}^0 = *$. We set the convention $\mathbb{D}^{-1} := \emptyset$.

Corollary 3.19. *For each dimension n, the space of automorphisms*

$$\mathsf{Aut}_{\mathcal{B}\mathsf{un}^{\mathsf{vfr}}}(\mathbb{D}^n) \ \simeq \ *$$

is contractible.

Proof. By definition, $\mathbb{D}^n = \mathsf{S}^{\mathsf{fr}}(\mathbb{D}^{n-1})$. The result follows from Lemma 3.17, by induction on n; the base case of $n = 0$ is clear.

□

Remark 3.20. The contractibility of $\mathsf{Aut}^{\mathsf{vfr}}(\mathbb{D}^n)$ of Corollary 3.19 is equivalent to the homotopy equivalence

$$\mathsf{Aut}_{\mathcal{B}\mathsf{un}}(\mathbb{D}^n) \simeq \mathsf{vfr}(\mathbb{D}^n)$$

between conically smooth diffeomorphisms of $\mathbb{D}^n$ and the space of vari-framings of $\mathbb{D}^n$. This statement about disk-stratified manifolds stands in stark constrast with the case of manifolds with boundary. For instance, consider D^n the compact smooth unit n-disk (which is not disk-stratified in our sense, since $\partial D^n \cong S^{n-1}$ is not disk-stratified). The space of diffeomorphisms $\mathsf{Aut}(D^n)$ is extremely complicated: its path components surject onto the group of exotic n-spheres by Kervaire–Milnor [KM] for $n > 4$; it has nontrivial higher homology by the calculations of Farrell–Hsiang [FS]. Consequently, the homotopy type of $\mathsf{Aut}(D^n)$ is not realizable in any familar way as a space of framings of D^n.

3.4. **Disks.** We introduce compact vari-framed *disk*-stratified spaces. We do so through a universal property, as the smallest collection containing $\emptyset$ and $*$ that is closed under the formation of framed suspension as well as *closed covers*, which we now define. Disk-stratified capture our intuition of 'finely stratified spaces', for they will ultimately play the role of a basis for a locality among stratified spaces concerning gluing along strata.

A two-term open cover of a smooth manifold determines a pushout diagram

$$\begin{tikzcd} M & V \arrow[l] \\ U \arrow[u] & U \cap V \arrow[l] \arrow[u] \end{tikzcd}$$

among smooth maps. In the case of stratified spaces we concern ourselves with likewise pushouts in which each arrow is not an open embedding but the inclusion of a closed union of strata. This can be phrased as a pullback diagram in $\mathcal{B}\mathsf{un}$, which we call a *purely closed cover*, using the monomorphism $(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl.inj}})^{\mathsf{op}} \underset{\text{Def 1.18}}{\hookrightarrow} \mathcal{B}\mathsf{un}$. The ∞-category $\mathcal{B}\mathsf{un}$ allows for another class of pullback diagram, which we call *refinement-closed covers*, which embody how a refinement of a stratum of a stratified space determines a refinement of that stratified space.

Definition 3.21 (Closed covers)**.** For each tangential structure τ, a limit diagram $[1]\times[1] \to \mathcal{B}\mathsf{un}^{\tau}$, written

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X'' \arrow[d] \\ X' \arrow[r] & X_0, \end{tikzcd}$$

is

- a *purely closed cover* if the diagram is comprised of closed morphisms;

- a *refinement-closed cover* if the horizontal arrows are refinement morphisms while the vertical arrows are closed morphisms.

The limit diagram is a *closed cover* if it is either a purely closed cover or a refinement-closed cover.

Remark 3.22. The opposite of a purely closed cover $([1]\times[1])^{\mathsf{op}} \to (\mathcal{B}\mathsf{un}^{\mathsf{cls}})^{\mathsf{op}} \simeq \mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,inj}}$ is a pushout diagram in $\mathcal{S}\mathsf{trat}$ by proper constructible embeddings. Through the results of §3 of [AFR], we conclude that the composite functor $([1]\times[1])^{\mathsf{op}} \to \mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,inj}} \xrightarrow{\mathsf{Exit}} \mathsf{Cat}_\infty$ is a pushout by fully faithful functors.

Definition 3.23. Let τ be a suspending tangential structure. The ∞-category of *compact disk-stratified τ-structured spaces* is the smallest full ∞-subcategory

$$\mathsf{c}\mathcal{D}\mathsf{isk}^\tau \ \subset\ \mathsf{c}\mathcal{B}\mathsf{un}^\tau$$

with the following properties.

(1) For each $i=-1,0$, and for each τ-structure g on $\mathbb{D}^i$, the object $(\mathbb{D}^i,g)$ belongs to $\mathsf{c}\mathcal{D}\mathsf{isk}^\tau$.
(2) The framed suspension $\mathsf{S}^{\mathsf{fr}}(X)$ belongs to $\mathsf{c}\mathcal{D}\mathsf{isk}^\tau$ whenever X does.
(3) An object $X\in\mathsf{c}\mathcal{B}\mathsf{un}^\tau$ belongs to $\mathsf{c}\mathcal{D}\mathsf{isk}^\tau$ whenever there is a closed cover in $\mathcal{B}\mathsf{un}^\tau$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X'' \arrow[d] \\ X' \arrow[r] & X_0 \end{tikzcd}$$

in which X' and X_0 and X'' belong to $\mathsf{c}\mathcal{D}\mathsf{isk}^\tau$.

For each dimension n, the ∞-category of *compact disk-stratified τ-structured n-manifolds* is the intersection

$$\mathsf{c}\mathcal{D}\mathsf{isk}^\tau_n \ :=\ \mathsf{c}\mathcal{D}\mathsf{isk}^\tau \underset{\mathcal{B}\mathsf{un}^\tau}{\times} \mathcal{B}\mathsf{un}^\tau_{\leq n}\ .$$

Remark 3.24. Definition 3.23 can be approached iteratively. For instance, (1) grants that $\mathbb{D}^0=*$ belongs to $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}$. Thereafter, (2) inductively grants that $\mathbb{D}^{n+1}:=\mathsf{S}^{\mathsf{fr}}(\mathbb{D}^n)$ belongs to $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}$ for each $n\geq 0$. Point (3) thereafter gives, in the case $X_0=\mathbb{D}^{-1}=\emptyset$ of a purely closed cover, that finite disjoint unions of $\mathbb{D}^n$'s belong to $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_n$. Also, for each $0\leq i\leq n$ the two standard closed morphisms $i_\pm\colon \mathbb{D}^n\to\mathbb{D}^i$ (visit Remark 3.22) give that the pullback $\mathbb{D}^n\underset{\mathbb{D}^i}{\times}\mathbb{D}^n$, which is the stratified space obtained by gluing together two n-disks along oppositely arranged i-hemispheres, belongs to $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}$. Iterating such purely closed covers, we see that pasting diagrams belong to $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}$, in particular. More complex stratified spaces are obtained through refinement-closed covers. Through these formations one sees that simplices, equipped with a particular vari-framing, belong to $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}$. This is illustrated as Figure 4. Thereafter, we see that triangulations of closed manifolds belong to $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}$, provided the triangulation is equipped with a vari-framing.

The 0-dimensional case is explicit, as the next result demonstrates.

Lemma 3.25. *There is an equivalence of ∞-categories*

$$\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{\leq 0} \ \simeq\ \mathsf{Fin}^{\mathsf{op}}$$

with the opposite of finite sets.

Proof. The idea of this proof is consistent of the main result of [AFR]: we show that both ∞-categories, when viewed as sheaves on stratified spaces, classify the same structure.

By inspection, there is a unique equivalence of the restricted vertical constructible vector bundles $\mathsf{T}^{\mathsf{v}}_{|\mathcal{B}\mathsf{un}_{\leq 0}} \simeq \epsilon^{\mathsf{vdim}}_{|\mathcal{B}\mathsf{un}_{\leq 0}}$. Therefore, the functor $\mathsf{vfr}_{\leq 0}\to\mathcal{E}\mathsf{xit}$ is an equivalence. It follows that the projection $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{\leq 0}\xrightarrow{\simeq}\mathsf{c}\mathcal{B}\mathsf{un}_{\leq 0}$ is an equivalence. The ∞-category $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{\leq 0}$ classifies proper constructible

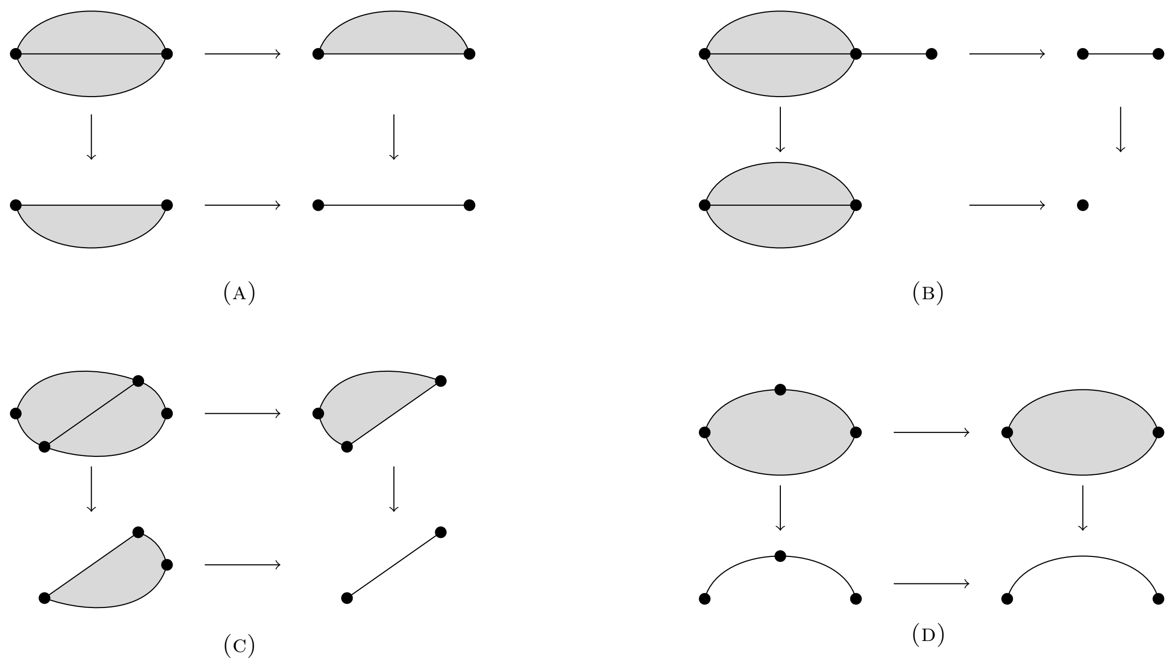

FIGURE 4. (A)–(C) depict purely closed covers; (D) depicts a refinement-closed cover.

bundles with finite fibers. Each such constructible bundle $E \to K$ has a unique path lifting property in Strat,

$$\begin{tikzcd} \Delta^{\{1\}} \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed, "\exists !"] & K, \end{tikzcd}$$

which implies the functor $\mathsf{Exit}(E) \to \mathsf{Exit}(K)$ is a right fibration whose fibers are 0-types. The latter is classified by a functor $\mathsf{Exit}(K) \to \mathsf{Fin}^{\mathsf{op}}$. By inspection, this assignment restricts along conically smooth maps $K' \to K$, and so we have the desired functor $\mathsf{c}\mathcal{B}\mathsf{un}_{\leq 0} \to \mathsf{Fin}^{\mathsf{op}}$.

Inspecting the case that $K = *$ reveals that this functor is essentially surjective. Inspecting the case that $K = \Delta^1$ reveals that this functor is bijective on mapping components. The above unique lifting property implies the morphism spaces of $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{\leq 0}$ are 0-types. We conclude that this functor $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{\leq 0} \to \mathsf{Fin}^{\mathsf{op}}$ is an equivalence of ∞-categories.

□

The 1-dimensional case is manageable as the next result indicates. This result is useful for comparison with more combinatorial entities, such as $\boldsymbol{\Delta}$.

Lemma 3.26. *The ∞-category $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_1$ is an ordinary category.*

Proof. We must show that, for each pair of objects D and D' in $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_1$, the space of morphisms $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_1(D, D')$ has contractible components. Recall the group ∞-category $\underline{\mathsf{O}}$ of Construction 2.26. By inspection, there is an identification of group ∞-categories $\underline{\mathsf{O}}_{|\leq 1} \simeq (\mathbb{Z}/2\mathbb{Z})^{\triangleleft}$. In particular, the projection map between spaces of morphisms

$$\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_1(D, D') \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_{\leq 1}(D, D')$$

has fibers which are 0-types; here we are using the same notation for objects in $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}$ and their unstructured projections to $\mathsf{c}\mathcal{D}\mathsf{isk}$. So it is enough to prove that $\mathsf{c}\mathcal{D}\mathsf{isk}_{\leq 1}(D, D')$ is a 0-type.

Let $s_0 \in S$ be a pointed connected smooth manifold, and consider an S-point of the striation sheaf $\mathsf{c}\mathcal{D}\mathsf{isk}_1(D, D')$. This classifies a proper constructible bundle $E \to S \times \Delta^1$ whose fibers are bounded above in dimension by 1, equipped with identifications $E_{|S\times\Delta^{\{0\}}} \simeq D \times S$ and $E_{|S\times\Delta^{\{1\}}} \simeq D' \times S$ over S. We wish to construct an equivalence $E \simeq E_{|\{s_0\}\times\Delta^1} \times S$ over $S \times \Delta^1$.

Consider the link system over S:

$$D \times S \cong E_{|S\times\Delta^{\{0\}}} \xleftarrow{\pi} \mathsf{Link}_{E_{|S\times\Delta^{\{0\}}}}(E) \xrightarrow{\gamma} E_{|S\times\Delta^{\{1\}}} \cong D' \times S\ ;$$

the map π is proper and constructible while the map γ is a refinement. In §6 of [AFR] we use flows to construct an isomorphism over $S \times \Delta^1$:

$$\Big(D \times S \coprod_{\mathsf{Link}_{E_{|S\times\Delta^{\{0\}}}}(E)\times\Delta^{\{0\}}} \mathsf{Link}_{E_{|S\times\Delta^{\{0\}}}}(E) \times \Delta^1 \coprod_{\mathsf{Link}_{E_{|S\times\Delta^{\{0\}}}}(E)\times\Delta^{\{1\}}} D' \times S\Big) \xrightarrow{\cong} E\ .$$

In other words, the given S-point of $\mathsf{c}\mathcal{D}\mathsf{isk}_1(D, D')$ admits a factorization as an S-point of $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{c.cr}}(D, D'')$ composed with an S-point of $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{ref}}(D'', D')$. Therefore, it is enough to consider these two cases separately.

Suppose the given S-point of $\mathsf{c}\mathcal{D}\mathsf{isk}_1(D, D')$ factors through $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{ref}}(D, D') \simeq \mathcal{S}\mathsf{trat}^{\mathsf{ref}}(D, D')$. By definition, the dimension of D' is bounded above by 1, and each connected stratum of D is either a singleton or an open interval. It follows from the connectivity of S that this S-point of $\mathcal{S}\mathsf{trat}^{\mathsf{ref}}(D, D')$ is constant, at $\gamma_{|\{s_0\}}$.

Suppose the given S-point of $\mathsf{c}\mathcal{D}\mathsf{isk}_1(D, D')$ factors through $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr,c.cr}}(D, D') \simeq \mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}(D', D)$. Denote by $D_0 \subset D$ the 0-dimensional strata; denote by $\{\overline{D}'_\alpha\}$ the collection of the closures of each 1-dimensional stratum of D', this collection is indexed by a finite set. As established in §7 of [AFT], there is the blow-up square in Strat

$$\begin{tikzcd}
\mathsf{Link}_{D'_0}(D') \arrow[r] \arrow[d] & \coprod_\alpha \overline{D}_\alpha \arrow[d] \\
D_0 \arrow[r] & D.
\end{tikzcd}$$

This diagram is a pushout by proper constructible maps. By definition, the dimension of D' is bounded above by 1, and the closure of each connected stratum of $\overline{D}_\alpha$ is either a singleton or a closed interval. It follows that $\mathsf{Link}_{D'_0}(D')$ is a finite set. We are therefore reduced to the case that D' is either a finite set or a closed interval.

Should D' be a finite set, then any constructible bundle $D' \to D$ factors through $D_0 \subset D$, the 0-dimensional strata of D. This reduces us to the situation that both D and D' are finite sets. The result in this case follows directly from Lemma 3.25.

Should D' be a closed interval, then any constructible bundle $D' \to D$ factors surjectively through a constructible closed subspace $D'' \subset D$ which is either a singleton or a closed interval. This reduces us to the situation that D is $*$ or $\mathbb{D}^1$. The result in the first case is trivially true. The result for the second case is true because any surjective constructible bundle $\mathbb{D}^1 \to \mathbb{D}^1$ is an isomorphism, the space of which is contractible.

□

The next result facilitates inductive arguments on dimension; we use it within a number of upcoming proofs.

Lemma 3.27. *For each dimension n there is a localization*

$$(-)_{\leq n}\colon\ \mathcal{B}\mathsf{un}^{\mathsf{vfr}} \rightleftarrows \mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$$

which retricts to a localization

$$(-)_{\leq n}\colon\ \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}} \rightleftarrows \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\ .$$

The units of these adjunctions are by closed morphisms.

Proof. As established in [AFT] (see §2), there is an n-skeleton functor $(-)_{\leq n}\colon \mathsf{Strat}^{\mathsf{emb}} \to \mathsf{Strat}^{\mathsf{emb}}$ on stratified spaces and open embeddings thereamong, given by removing strata of dimension greater than n. Taking this n-skeleton fiberwise defines a functor $(-)_{\leq n} : \mathsf{Bun} \to \mathsf{Bun}$ over Strat. Namely, assign to a constructible bundle $M \xrightarrow{\pi} K$ the n-skeleton $(M|K)_{\leq n}$ relative K, which we define as the stratified subspace of M consisting of those $x \in M$ for which there is a bound of local dimensions of the fiber $\mathsf{dim}_x(\pi^{-1}\pi(x)) \leq n$. The restriction $(M|K)_{\leq n} \xrightarrow{\pi_|} K$ is again a constructible bundle.

Applying the topologizing diagram (§2 from [AFR]) to this functor $(-)_{\leq n} : \mathsf{Bun} \to \mathsf{Bun}$ gives a functor of ∞-categories $(-)_{\leq n} : \mathcal{B}\mathsf{un} \to \mathcal{B}\mathsf{un}$. By construction, this functor factors through the ∞-subcategory $\mathcal{B}\mathsf{un}_{\leq n}$. We now construct a unit natural transformation $\mathsf{id} \to (-)_{\leq n}$. We do this by applying the topologizing diagram to an endofunctor $\mathsf{Bun} \to \mathsf{Bun}$ over $\mathsf{Strat} \xrightarrow{-\times\Delta^1} \mathsf{Strat}$ with the property that over $\Delta^{\{0\}}$ this endofunctor is the identity functor and over $\Delta^{\{1\}}$ it is the functor $(-)_{\leq n}$. This endofunctor is, for each stratified space K, the assignment of groupoids of constructible bundles

$$(M \xrightarrow{\pi} K) \ \mapsto \ \Big(M \coprod_{(M|K)_{\leq n}\times\Delta^{\{0\}}} (M|K)_{\leq n} \times \Delta^1 \xrightarrow{\pi_| \times \mathsf{id}_{\Delta^1}} K \times \Delta^1\Big) \ ;$$

this assignment evidently pulls back along maps among the K-argument. This assignment has the requisite restrictions over $\Delta^{\{0\}}$ and $\Delta^{\{1\}}$. By construction, this unit is by closed morphisms.

We extend this to the general τ-structured case. We do this by dint of the applying requirement which ensures that the restriction of $\mathcal{B}\mathsf{un}^\tau \to \mathcal{B}\mathsf{un}$ to closed morphisms is a coCartesian fibration, applied to fact that the unit of the above localization is implemented by closed morphisms. Picture this geometrically as follows. For each K-point of $\mathcal{B}\mathsf{un}^\tau$ represented by a constructible bundle $M \to K$, we can again form the stratified space $M \times \{0\} \cup (M \times \Delta^1 | K \times \Delta^1)_{\leq n}$. By the coCartesian property, a τ-structure on $M \times \{0\}$ canonically extends to a fiberwise τ-structure on the entire space over $K \times \Delta^1$. Formally, we construct the corresponding correspondence of ∞-category $\widetilde{\mathcal{B}\mathsf{un}^\tau} \to \widetilde{\mathcal{B}\mathsf{un}} \to [1]$, and we show the composite functor is both a Cartesian and a coCartesian fibration. First, it is again manifestly Cartesian. To check the coCartesian property amounts to the existence of coCartesian morphisms with any fixed source $M \in \mathcal{B}\mathsf{un}^\tau \simeq \widetilde{\mathcal{B}\mathsf{un}^\tau}_{|\{0\}}$ lying over the single non-identity morphism in [1]. We first choose the lift in $\widetilde{\mathcal{B}\mathsf{un}}$, using that $\widetilde{\mathcal{B}\mathsf{un}} \to [1]$ is coCartesian. This lift is a closed morphism, by construction, and $\mathcal{B}\mathsf{un}^\tau \to \mathcal{B}\mathsf{un}$ is a coCartesian fibration over closed morphisms; consequently, a second lift can be chosen, and we obtain that the composite functor $\widetilde{\mathcal{B}\mathsf{un}^\tau} \to [1]$ is coCartesian and Cartesian.

Lastly, we set $\tau = \mathsf{vfr}$ and observe the first localization. Because $(M|K)_{\leq n} \to K$ is proper whenever the constructible bundle $M \to K$ is proper, the first localization restricts as a localization $\mathsf{c}\mathcal{B}\mathsf{un}^{\mathsf{vfr}} \rightleftarrows \mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$. Finally, by construction $M_{\leq n}$ is disk-stratified whenever the vari-framed stratified space M is disk-stratified. This completes the result. □

The next result pushes the contractibility of framed suspensions (Corollary 3.19).

Theorem 3.28 (Discreteness of automorphisms)**.** *For each compact vari-framed disk-stratified space M, the space of automorphisms $\mathsf{Aut}_{\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}}(M)$ has contractible components.*

Proof. We proceed by induction on the dimension n of M. The base case of the induction, where $n = 0$ and so M is the datum of a finite set, follows from Lemma 3.25 which identifies $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}(M, M)$ as the space of maps of finite sets from M to M.

We assume the statement for manifolds up to dimension $n-1$. Using that the $(n-1)$-skeleon defines a functor (Lemma 3.27), there is a map of spaces of automorphisms

$$(-)_{<n}\colon\ \mathsf{Aut}_{\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}}(M) \longrightarrow \mathsf{Aut}_{\mathsf{c}\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{vfr}}}(M_{<n}) \ .$$

By induction, it is therefore enough to prove that each fiber of this map has contractible components. The codomain of this map being a group, all fibers have the same homotopy type. So it suffices to

show that the fiber over the identity morphism of $M_{<n}$ has contractible components. We write this fiber as $\mathsf{Aut}(M, M_{<n})$.

Consider the collection of closed morphisms $\{M \xrightarrow{\mathsf{cls}} D_\alpha\}$ indexed by the closures of the connected n-dimensional strata of M. Because M lies in $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$, each D_α is a refinement of $\mathbb{D}^n$. Also, M belonging to $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ implies the canonical diagram in $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \prod\limits_\alpha D_\alpha \arrow[d] \\ M_{<n} \arrow[r] & \prod\limits_\alpha (D_\alpha)_{<n} \end{tikzcd}$$

is a limit diagram. In particular, the canonical map of spaces

$$\mathsf{Aut}(M, M_{<n}) \longrightarrow \prod_\alpha \mathsf{Aut}\big(D_\alpha, (D_\alpha)_{<n}\big)$$

is an equivalence. We are reduced to the case where $M = D_\alpha$ is a refinement of $\mathbb{D}^n$.

By design, each D_α is the limit in $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ of a refinement-closed cover

$$\begin{tikzcd} D_\alpha \arrow[r] \arrow[d] & \mathbb{D}^n \arrow[d] \\ (D_\alpha)_{<n} \arrow[r] & \partial\mathbb{D}^n. \end{tikzcd}$$

This implies that the natural map of fibers $\mathsf{Aut}\big(D_\alpha, (D_\alpha)_{<n}\big) \to \mathsf{Aut}(\mathbb{D}^n, \partial\mathbb{D}^n)$ is an equivalence. Thus, we are reduced to the case $M = \mathbb{D}^n$. For this case, the discreteness of $\mathsf{Aut}(\mathbb{D}^n, \partial\mathbb{D}^n)$ follows from the discreteness of $\mathsf{Aut}(\mathbb{D}^n)$ proved in Corollary 3.19 and the discreteness of $\mathsf{Aut}(\partial\mathbb{D}^n)$ posited in the induction.

□

3.5. **Wreath.** We compare the *wreath* construction, which ultimately defines $\mathbf{\Theta}_n$, with iterated constructible bundles.

We recall the wreath construction. We use the notation $\mathsf{Fin}_* := \mathsf{Fin}^{*/}$ for the ∞-category of based finite sets, an object of which is a typically denoted as $I_+ = I \amalg \{+\}$ so that the base-point is visible. We use the notation

$$\mathsf{Fin}_{*\star} \;\subset\; \mathsf{Fin}_*^{\star_+/}$$

for the full ∞-subcategory of the under ∞-category consisting of those based maps $(\star_+ \xrightarrow{f} I_+)$ from the two-element based set that satisfy the inequality $f(\star) \neq +$. Note the standard projection functor $\mathsf{Fin}_{*\star} \to \mathsf{Fin}_*$.

After presentability considerations, Lemma 5.19 of the appendix gives that base change along $\mathsf{Fin}_{*\star} \to \mathsf{Fin}_*$ is a left adjoint.

Definition 3.29 (Wreath)**.** For each ∞-category $\mathcal{D} \to \mathsf{Fin}_*$ over based finite sets, the *wreath* functor

$$\mathcal{D} \wr - \colon\; \mathsf{Cat}_\infty \longrightarrow \mathsf{Cat}_{\infty/\mathcal{D}}$$

is the right adjoint to the composite functor

$$\mathsf{Cat}_{\infty/\mathcal{D}} \longrightarrow \mathsf{Cat}_{\infty/\mathsf{Fin}_*} \xrightarrow{-\underset{\mathsf{Fin}_*}{\times}\mathsf{Fin}_{*\star}} \mathsf{Cat}_{\infty/\mathsf{Fin}_{*\star}} \longrightarrow \mathsf{Cat}_\infty \quad .$$

Example 3.30. We regard the opposite of the simplex category $\mathbf{\Delta}^{\mathsf{op}}$ as an ∞-category over based finite sets by way of the simplicial circle

$$\Delta[1]/\partial\Delta[1] \colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Fin}_* \quad .$$

Observation 3.31. For each ∞-category $\mathcal{D}$ over Fin_*, a fully faithful functor $\mathcal{C} \hookrightarrow \mathcal{C}'$ between ∞-categories determines a fully faithful functor $\mathcal{D} \wr \mathcal{C} \to \mathcal{D} \wr \mathcal{C}'$.

Lemma 3.32. *For each dimension n, and each suspending tangential structure τ, there is a functor*

$$\mathsf{c}\mathcal{D}\mathsf{isk}_n^\tau \longrightarrow \mathsf{Fin}_* ~.$$

Proof. It is enough to prove the result for the case $\tau \xrightarrow{\simeq} \mathcal{E}\mathsf{xit}$ is an equivalence. Consider the full ∞-subcategory $\mathcal{E}\mathsf{xit}^{=n}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \subset \mathcal{E}\mathsf{xit}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n}$ consisting of those conically smooth maps $* \to D \in \mathsf{c}\mathcal{D}\mathsf{isk}_n$ that factor through $D_n \subset D$, the n-dimensional strata of D. Using that each stratum of an object of $\mathsf{c}\mathcal{D}\mathsf{isk}_n^\tau$ is contractible, the projection

$$\mathcal{E}\mathsf{xit}^{=n}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n$$

is classified by a functor $\mathsf{c}\mathcal{D}\mathsf{isk}_n \longrightarrow \mathsf{Fin}_*$.

□

Lemma 3.33. *Let n be a dimension, and let τ be a suspending tangential structure. For each ∞-category $\mathcal{C}$ there is a functor*

$$\mathsf{c}\mathcal{D}\mathsf{isk}_n^\tau \wr \mathcal{C} \longleftarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^\tau \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathcal{C}} ~.$$

This functor admits a fully faithful left adjoint if and only if $\mathcal{C}$ has an initial object.

Proof. It is enough to prove the result for the case $\tau \xrightarrow{\simeq} \mathcal{E}\mathsf{xit}$ is an equivalence. Consider the full ∞-subcategory $\mathcal{E}\mathsf{xit}^{=n}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \subset \mathcal{E}\mathsf{xit}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n}$ from the explanation of Lemma 3.32. Using that each stratum of an object of $\mathsf{c}\mathcal{D}\mathsf{isk}_n$ is contractible, we identify the fiberwise path components

$$\pi_0^{\mathsf{fib}}\colon~ \mathcal{E}\mathsf{xit}^{=n}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n} ~\xrightarrow{\simeq}~ \mathsf{Fin}_{*\star|\mathsf{c}\mathcal{D}\mathsf{isk}_n}$$

over $\mathsf{c}\mathcal{D}\mathsf{isk}_n$. We obtain a fully faithful functor

$$\mathsf{Fin}_{*\star|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \longrightarrow \mathcal{E}\mathsf{xit}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \tag{12}$$

over $\mathsf{c}\mathcal{D}\mathsf{isk}_n$. In particular, for each ∞-category $\mathcal{C}$ there is a restriction functor

$$\mathsf{Fun}(\mathcal{E}\mathsf{xit}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n}, \mathcal{C}) \longrightarrow \mathsf{Fun}(\mathsf{Fin}_{*\star|\mathsf{c}\mathcal{D}\mathsf{isk}_n}, \mathcal{C}) ~. \tag{13}$$

This is the desired functor, as seen by unwinding the constructions $\mathcal{C} \mapsto \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathcal{C}}$ and $\mathcal{C} \mapsto \mathsf{c}\mathcal{D}\mathsf{isk}_n \wr \mathcal{C}$ as right adjoints.

Now, the fully faithfulness of (12) gives that a left adjoint to (13) is fully faithful, whenever one exists. Any such left adjoint to (13) is computed via left Kan extension:

$$\big(\mathsf{Fin}_{*\star|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \xrightarrow{\mathcal{F}} \mathcal{C}\big) \mapsto \Big(\mathcal{E}\mathsf{xit}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \ni (x \in D) \mapsto \underset{(x' \in D') \in (\mathcal{E}\mathsf{xit}^{=n}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n})_{/(x\in D)}}{\mathsf{colim}} \mathcal{F}(x' \in D'_n)\Big) ~.$$

The full ∞-subcategory $\mathcal{E}\mathsf{xit}^{=n}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n} \subset \mathcal{E}\mathsf{xit}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n}$ has the property that, for each object $(x \in D) \in \mathcal{E}\mathsf{xit}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n}$, the over ∞-category indexing this colimit

$$(\mathcal{E}\mathsf{xit}^{=n}_{|\mathsf{c}\mathcal{D}\mathsf{isk}_n})_{/(x\in D)}$$

is either empty or has a final object. Indeed, each stratum of an object of $\mathsf{c}\mathcal{D}\mathsf{isk}_n$ is contractible and has dimension bounded by n. We conclude that such a left adjoint exists if and only if $\mathcal{C}$ has an initial object.

□

Corollary 3.34. *For each pair of dimension i and j, and for each suspending tangential structure $\tau \to \mathcal{E}\mathsf{xit}$ for which the restricted projection $\tau_{\leq 0} \xrightarrow{\simeq} \mathcal{E}\mathsf{xit}_{\leq 0}$ is an equivalence, there is a fully faithful functor*

$$\mathsf{c}\mathcal{D}\mathsf{isk}_i^\tau \wr \mathsf{c}\mathcal{D}\mathsf{isk}_j^\tau ~\hookrightarrow~ \mathsf{c}\mathcal{D}\mathsf{isk}_i^\tau \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{D}\mathsf{isk}_j^\tau} ~.$$

In particular, there are fully faithful functors

$$\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}} \wr \mathsf{c}\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{vfr}} ~\hookrightarrow~ \mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{vfr}}}$$

and

$$\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{sfr}} \wr \mathsf{c}\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{sfr}} \hookrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{sfr}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{sfr}}} .$$

Proof. After Lemma 3.33 it need only be checked that $\mathsf{c}\mathcal{D}\mathsf{isk}_j^{\tau}$ has an initial object. Because of the condition on $\tau_{\leq 0}$, this is the case exactly because $\mathsf{c}\mathcal{D}\mathsf{isk}_j$ has an initial object, which is $*$.

□

3.6. **Iterated linear orders.** We recall the definition of Joyal's category $\mathbf{\Theta}_n$ as well as some notions within it. We first recall the following notions within the simplex category $\mathbf{\Delta}$.

Definition 3.35.

- **Inerts:** The *inert* subcategory $\mathbf{\Delta}_{\mathsf{inrt}} \subset \mathbf{\Delta}$ consists of the same objects and those morphisms $[p] \xrightarrow{\rho} [q]$ for which $\rho(i-1) = \rho(i) - 1$ for each $0 < i \leq p$.
- **Actives:** The *active* subcategory $\mathbf{\Delta}_{\mathsf{act}} \subset \mathbf{\Delta}$ consists of the same objects and those morphisms $[p] \xrightarrow{\rho} [q]$ that preserve extrema: $\rho(0) = 0$ and $\rho(p) = q$.
- **Segal covering diagrams:** A colimit diagram $[1] \times [1] \to \mathbf{\Delta}$, written

$$\begin{tikzcd} {[p_0]} \arrow[r] \arrow[d] & {[p']} \arrow[d] \\ {[p'']} \arrow[r] & {[p],} \end{tikzcd}$$

  is a *Segal covering* diagram if each arrow is inert.
- **Univalence diagram:** The *univalence* diagram is the colimit diagram in $\mathbf{\Delta}$

$$\begin{tikzcd} & \{1<3\} \arrow[r] \arrow[d] & * \arrow[dd] \\ \{0<2\} \arrow[r] \arrow[d] & \{0<1<2<3\} \arrow[dr] & \\ * \arrow[rr] & & *. \end{tikzcd}$$

Remark 3.36. Consider the subcategory $\mathsf{Fin}_{\mathsf{inj}} \subset \mathsf{Fin}$ consisting of the same objects and those morphisms which are *injective*. There is a monomorphism $\mathsf{Fin}_{\mathsf{inj}}^{\mathsf{op}} \to \mathsf{Fin}_*$ given by 1-point compactifications and collapse-maps thereamong. Recall from Example 3.30 the functor $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Fin}_*$. With respect to these functors, there is an identification $\mathbf{\Delta}_{\mathsf{inrt}}^{\mathsf{op}} \simeq \mathbf{\Delta}^{\mathsf{op}} \underset{\mathsf{Fin}_*}{\times} \mathsf{Fin}_{\mathsf{inj}}^{\mathsf{op}}$.

Remark 3.37. It is standard that $(\mathbf{\Delta}_{\mathsf{act}}, \mathbf{\Delta}_{\mathsf{inrt}})$ is a factorization system on $\mathbf{\Delta}$.

The next result is definitional.

Proposition 3.38. *The standard functor*

$$\mathbf{\Delta} \xrightarrow{[\bullet]} \mathsf{Cat}_\infty$$

carries both Segal covering diagrams and the univalence diagram to a colimit diagram.

Proof. Let $[1] \times [1] \to \mathbf{\Delta}$, written

$$\begin{tikzcd} {[p_0]} \arrow[r] \arrow[d] & {[p']} \arrow[d] \\ {[p'']} \arrow[r] & {[p],} \end{tikzcd}$$

be a Segal covering diagram. Because colimits commute with one another, to argue that this is a colimit diagram among ∞-categories it is enough to argue the case that $[p''] = \{0 < 1\}$ and $[p] = \{1\}$ and $[p'] = \{1 < \dots, p\}$. That is, for each ∞-category $\mathcal{C}$, the canonical map of spaces of functors

$$\mathsf{Map}\big([p], \mathcal{C}\big) \longrightarrow \mathsf{Map}\big(\{0<1\}, \mathcal{C}\big) \underset{\mathsf{Map}(\{1\},\mathcal{C})}{\times} \mathsf{Map}\big(\{1 < \dots < p\}, \mathcal{C}\big)$$

must be an equivalence. This is manifestly the case.

Consider the functor $\mathcal{E}^{\triangleright} \to \mathbf{\Delta}$ representing the univalence diagram. Consider the ∞-category $E := \mathsf{colim}(\mathcal{E} \to \mathbf{\Delta} \to \mathsf{Cat}_\infty)$. This ∞-category corepresents the data of a morphism together with a left and a right inverse:

$$\mathsf{Map}_{\mathsf{Cat}_\infty}(E, \mathcal{C}) \ \simeq\ \Big\{ d \xrightarrow{f^R} c \xrightarrow{f} d \xrightarrow{f^L} c \text{ and } f \circ f^R \simeq \mathsf{id}_d \text{ and } \mathsf{id}_c \simeq f^L \circ f, \text{ all in } \mathcal{C} \Big\}\ .$$

Manifestly, left and right inverses are unique whenever they exist. Thus, we identify the above space simply as $\mathsf{Map}\big([1], \mathcal{C}^\sim\big)$, the space of morphisms in the maximal ∞-subgroupoid of $\mathcal{C}$. There is the further identification $\mathcal{C}^\sim \xrightarrow{\simeq} \mathsf{Map}\big([1], \mathcal{C}^\sim\big)$ induced by the unique functor $[1] \to *$. In other words, the unique functor $E \to *$ is an equivalence of ∞-categories. In conclusion, the composite functor $\mathcal{E}^{\triangleright} \to \mathbf{\Delta} \to \mathsf{Cat}_\infty$ is a colimit diagram.

□

We give a definition of Joyal's category $\mathbf{\Theta}_n$ ([Jo2]). This follows Definition 3.9 in [Be], adapted through the wreath Construction 2.4.4.1 of [Lu2].

Definition 3.39 ($\mathbf{\Theta}_n$)**.** For $n \geq 0$, the ∞-category over $\mathbf{\Delta}^{\mathsf{op}}$,

$$\mathbf{\Theta}_n^{\mathsf{op}} \longrightarrow \mathbf{\Delta}^{\mathsf{op}}\ ,$$

is defined inductively as $\mathbf{\Theta}_n^{\mathsf{op}} := \mathbf{\Delta}^{\mathsf{op}} \wr \mathbf{\Theta}_{n-1}^{\mathsf{op}}$ for $n > 0$, while $\mathbf{\Theta}_0^{\mathsf{op}} := * \xrightarrow{\{[0]\}} \mathbf{\Delta}^{\mathsf{op}}$.

Remark 3.40. We have presented $\mathbf{\Theta}_n^{\mathsf{op}}$ as an ∞-category. However, for each pair of objects $T, T' \in \mathbf{\Theta}_n^{\mathsf{op}}$, the space of morphisms $\mathbf{\Theta}_n^{\mathsf{op}}(T', T)$ is a 0-type; in other words, $\mathbf{\Theta}_n^{\mathsf{op}}$ is an ordinary category. Indeed, this follows quickly by induction on n, using that each of $\mathbf{\Delta}^{\mathsf{op}}$ and Fin_* and $\mathsf{Fin}_{*\star}$ are ordinary categories.

We record some notions within the category $\mathbf{\Theta}_n := (\mathbf{\Theta}_n^{\mathsf{op}})^{\mathsf{op}}$.

Definition 3.41. Let $n \geq 0$.

- **Inerts:** The *inert* ∞-subcategory of $\mathbf{\Theta}_n^{\mathsf{op}}$ is defined inductively as $\mathbf{\Theta}_{n,\mathsf{inrt}}^{\mathsf{op}} := \mathbf{\Delta}_{\mathsf{inrt}}^{\mathsf{op}} \wr \mathbf{\Theta}_{n-1,\mathsf{inrt}}^{\mathsf{op}}$ for $n > 0$ while $\mathbf{\Theta}_{0,\mathsf{inrt}} = \mathbf{\Theta}_0 = *$.
- **Actives:** The *active* ∞-subcategory of $\mathbf{\Theta}_n^{\mathsf{op}}$ is defined inductively as $\mathbf{\Theta}_{n,\mathsf{act}}^{\mathsf{op}} := \mathbf{\Delta}_{\mathsf{act}}^{\mathsf{op}} \wr \mathbf{\Theta}_{n-1,\mathsf{act}}^{\mathsf{op}}$ for $n > 0$ while $\mathbf{\Theta}_{0,\mathsf{act}} = \mathbf{\Theta}_0 = *$.
- **Cells:** For each $0 \leq i \leq n$, the *i-cell* $c_i \in \mathbf{\Theta}_n^{\mathsf{op}}$ is the initial object if $i = 0$ and if $i > 0$ it is the object $* \xrightarrow{\{c_i\}} \mathbf{\Theta}_n^{\mathsf{op}}$ representing the pair of functors $* \xrightarrow{\{[1]\}} \mathbf{\Delta}^{\mathsf{op}}$ and $* \simeq \{[1]\} \underset{\mathsf{Fin}_*}{\times} \mathsf{Fin}_{*\star} \xrightarrow{\{c_{i-1}\}} \mathbf{\Theta}_{n-1}^{\mathsf{op}}$.
- **Segal covering diagrams:** A colimit diagram $[1] \times [1] \to \mathbf{\Theta}_n$, written

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d] & T' \arrow[d] \\ T'' \arrow[r] & T, \end{tikzcd}$$

  is a *Segal covering* diagram if each arrow is inert.
- **Univalence diagrams:** For $n > 0$, a colimit diagram $\mathcal{E}^{\triangleright} \to \mathbf{\Theta}_n$ is a *univalence* diagram if it has either of the following two properties:

– The projection $\mathcal{E}^{\triangleright} \to \mathbf{\Delta}$ factors through the functor $* \xrightarrow{\{c_1\}} \mathbf{\Delta}$, and the functor $(\mathcal{E}^{\triangleright})^{\mathsf{op}} \simeq (\mathcal{E}^{\triangleright})^{\mathsf{op}} \underset{\mathsf{Fin}_*}{\times} \mathsf{Fin}_{*\star} \to \mathbf{\Theta}_{n-1}^{\mathsf{op}}$ is the opposite of a univalence diagram.

– The projection $\mathcal{E}^{\triangleright} \to \mathbf{\Delta}$ is the univalence diagram, and the functor $(\mathcal{E}^{\triangleright})^{\mathsf{op}} \underset{\mathsf{Fin}_*}{\times} \mathsf{Fin}_{*\star} \to \mathbf{\Theta}_{n-1}^{\mathsf{op}}$ factors through $* \xrightarrow{\{c_0\}} \mathbf{\Theta}_{n-1}^{\mathsf{op}}$.

Remark 3.42. The active-inert factorization system on $\mathbf{\Delta}$ (see Remark 3.37) determines an active-inert factorization system on $\mathbf{\Theta}_n$ for each $n \geq 0$.

Observation 3.43. We note that, for each $0 \leq k \leq n$, there are monomorphisms among categories

$$\iota_k \colon \mathbf{\Theta}_k^{\mathsf{op}} \hookrightarrow \mathbf{\Theta}_n^{\mathsf{op}} \hookleftarrow \mathbf{\Theta}_{n-k}^{\mathsf{op}} \colon c_k \wr - ,$$

the first of which is fully faithful; these functors preserve Segal covering diagrams and univalence diagrams. For $k = 0$ the left functor is the inclusion of the initial object while the right functor is the identity. For $0 < k \leq n$ these functors are given through induction as

$$\mathbf{\Theta}_k^{\mathsf{op}} := \mathbf{\Delta}^{\mathsf{op}} \wr \mathbf{\Theta}_{k-1}^{\mathsf{op}} \overset{\mathsf{id}_{\mathbf{\Delta}^{\mathsf{op}}} \wr \iota_k}{\hookrightarrow} \mathbf{\Delta}^{\mathsf{op}} \wr \mathbf{\Theta}_{n-1}^{\mathsf{op}} =: \mathbf{\Theta}_n^{\mathsf{op}} := \mathbf{\Delta}^{\mathsf{op}} \wr \mathbf{\Theta}_{n-1}^{\mathsf{op}} \overset{\{[1]\} \wr (c_{k-1} \wr -)}{\hookleftarrow} \{\star_+\} \wr \mathbf{\Theta}_{n-k}^{\mathsf{op}} \simeq \mathbf{\Theta}_{n-k}^{\mathsf{op}} .$$

The injectivity assertions follow quickly by induction, for which it is useful that all ∞-categories here are ordinary categories. That these functoris preserve Segal covering diagrams follows because, by induction, they preserve inert morphisms, and, by induction, they preserve colimits. That these functors preserve univalence diagrams is direct from definitions.

3.7. **Cellular realization.** We define a fully faithful functor $\mathbf{\Theta}_n^{\mathsf{op}} \hookrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$, which we call *cellular realization*. The existence of this functor, and that it is fully faithful, is the culmination of choices behind the definition of $\mathcal{B}\mathsf{un}$ and its vari-framed version. This fully faithful functor founds the contact between higher categories and vari-framed differential topology, as we articulate as factorization homology of the coming section.

The next result establishes the cellular realization functor for dimension 1. Recall that a morphism in $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}$ is *closed/active* if it is carried to a closed/active morphism by the forgetful functor $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}} \to \mathsf{c}\mathcal{B}\mathsf{un}$.

Lemma 3.44. *There is a functor*

$$\langle - \rangle \colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}$$

with the following properties.

(1) *The functor is fully faithful.*
(2) *The functor carries $[0]$ to $\mathbb{D}^0$ and $[1]$ to $\mathbb{D}^1$.*
(3) *The functor carries the opposite of inert morphisms to closed morphisms.*
(4) *The functor carries the opposite of active morphisms to active morphisms.*
(5) *The functor carries Segal covers to purely closed covers.*
(6) *Let $[p] \xrightarrow{\rho} [q]$ be a morphism in $\mathbf{\Delta}$. For each factorization $\langle \rho \rangle \colon \langle [q] \rangle \xrightarrow{c} D \to \langle [p] \rangle$ in $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}$ in which c is a closed morphism, the object D belongs to the essential image of $\mathbf{\Delta}^{\mathsf{op}}$.*

Proof. Consider the full ∞-subcategory $\mathcal{D} \subset \mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}$ consisting of those D for which there is a refinement morphism $D \to \mathbb{D}^1$ or $D \to \mathbb{D}^0$ in $\mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}$. By inspection, this full ∞-subcategory has the likewise property as (6): should a morphism $D' \to D''$ in $\mathcal{D}$ factor through a closed morphism $D' \to D$ then D belongs to $\mathcal{D}$.

We will construct an equivalence of ∞-categories

$$\mathcal{D} \xrightarrow{\simeq} \mathbf{\Delta}^{\mathsf{op}}$$

as striation sheaves, according to the main result of [AFR]. Let K be a stratified space. Because the morphism spaces in $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ are 0-types (Lemma 3.26), the space of K-points of each of these striation sheaves is a 0-type.

Consider a functor $\mathsf{Exit}(K) \xrightarrow{(X\to K,\phi)} \mathcal{D}$. Denote the fiberwise 0-dimensional strata as $X_0 \to K$, and likewise $X_{>0} \to K$ for its complement. Choose a conically smooth embedding

$$e\colon X \hookrightarrow \mathbb{R}\times K$$

over K for which the pullback vertical vari-framing $e^*_{>0}\partial_t\colon \mathsf{Exit}(X_{>0}) \to \mathsf{vfr}$ is in the same component of the restriction $\phi_{>0}\colon \mathsf{Exit}(X_{>0}) \to \mathsf{vfr}$. This embedding e determines a fiberwise linear order $\leq$ on the constructible bundle $X_0 \to K$, by which we mean a constructible closed subspace of the fiber product $X_0 \underset{K}{\times} X_0$ that intersects each fiber $X_{|\{k\}} \times X_{|\{k\}}$ as a linear order. This subspace $\leq$ is constructible and closed, and does not depend on the choice of embedding e. Taking values in $\mathcal{D}$ implies $X_0 \to K$ is a surjective proper constructible bundle and has the following unique path lifting property in Strat

$$\begin{tikzcd} \Delta^{\{1\}} \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed, "\exists!"] & K. \end{tikzcd}$$

Furthermore, the constructible fiberwise linear order enhances the right fibration $\mathsf{Exit}(X_0) \to \mathsf{Exit}(K)$ to a Cartesian fibration whose fibers are non-empty finite linearly ordered sets. Such a Cartesian fibration is classified by a functor $\mathsf{Exit}(K) \xrightarrow{(X_0\to K,\leq)} \mathbf{\Delta}^{\mathsf{op}}$.

We have thus produced a well-defined assignment of K-points from those of $\mathcal{D}$ to those of $\mathbf{\Delta}^{\mathsf{op}}$. Tracing through the construction of this assignment, it restricts along conically smooth maps $K' \to K$, thereby producing the desired functor

$$\mathcal{D} \longrightarrow \mathbf{\Delta}^{\mathsf{op}}~.$$

We now wish to show this functor is an equivalence of ∞-categories. It is clearly essentially surjective, and surjective on mapping components. To see that this functor is injective on mapping components follows upon observing that, in the situation of the preceding argument, the space of embeddings $e\colon X \hookrightarrow \mathbb{R}\times K$ over K is connected. This proves property (1).

Properties (2),(3),(4), and (5) follow by direct inspection of the functor $\mathcal{D} \to \mathbf{\Delta}^{\mathsf{op}}$ just constructed. □

Remark 3.45. The value $\langle[0]\rangle$ is a $*$, regarded as a vari-framed manifold. For $p>0$, the value $\langle[p]\rangle$ is a var-framed refinement of the interval $\mathbb{D}^1=[-1,1]$ whose 0-dimensional strata bijective with the set $\{0,\dots,p\}$.

The definition of cellular realization is supported by the inductive Definition 3.39 of $\mathbf{\Theta}_n$ combined with Corollary 3.34, which compares the wreath construction to iterated constructible bundles, and the vari-framed lift of $\mathcal{B}\mathsf{un}^{\mathcal{B}\mathsf{un}} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$ provided by Corollary 3.11.

Definition 3.46 (Cellular realization)**.** For each dimension n, the *cellular realization* functor

$$\langle - \rangle\colon~ \mathbf{\Theta}_n^{\mathsf{op}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$$

is the composition

$$\langle - \rangle\colon~ \mathbf{\Theta}_n^{\mathsf{op}} ~:=~ \mathbf{\Delta}^{\mathsf{op}} \wr \mathbf{\Theta}_{n-1}^{\mathsf{op}} \underset{\text{induction}}{\longrightarrow} \mathbf{\Delta}^{\mathsf{op}} \wr \mathsf{c}\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{vfr}} \underset{\text{Cor } 3.34}{\longrightarrow} \mathbf{\Delta}^{\mathsf{op}} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{vfr}}} \underset{\text{Cor } 3.11}{\xrightarrow{\circ}} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$$

provided $n>1$. For $n=0$ it is the functor $* \xrightarrow{\{\mathbb{D}^0\}} \mathsf{c}\mathcal{D}\mathsf{isk}_0^{\mathsf{vfr}}$; for $n=1$ it is the functor of Lemma 3.44.

Remark 3.47. Intuitively, the value $\langle T\rangle$ is literally the pasting diagram associated to $T\in\mathbf{\Theta}_n$ which can be regarded as a stratified subspace of $\mathbb{R}^n$ as it inherits a vari-framing. To make this intuition precise and assemble this description as a functor in any point-set sense is not practical, if possible at all. The functor $\langle - \rangle$ fully embraces the setting of ∞-categories.

Observation 3.48. For each $0 \leq k \leq n$ the fully faithful functor $\iota_k \colon \mathbf{\Theta}_k^{\mathsf{op}} \hookrightarrow \mathbf{\Theta}_n^{\mathsf{op}}$ lies over $\mathsf{c}\mathcal{D}\mathsf{isk}_k^{\mathsf{vfr}} \hookrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$. This follows by induction on i from Observation 3.31, with base case the fully faithful functor $* \hookrightarrow \mathbf{\Delta}^{\mathsf{op}}$.

Recall the ∞-category $\mathsf{c}\mathcal{B}\mathsf{un}^n$ of §3.1 that classifies n-fold proper constructible bundles among stratified spaces. There is a τ-structured version of this ∞-category that we highlight as the following notation.

Notation 3.49. For τ a tangential structure, we inductively denote

$$\mathsf{c}\mathcal{B}\mathsf{un}^{\tau,n} ~:=~ \mathsf{c}\mathcal{B}\mathsf{un}^{\tau} \underset{\mathcal{B}\mathsf{un}}{\times} \mathcal{B}\mathsf{un}^{\mathsf{c}\mathcal{B}\mathsf{un}^{\tau,n-1}}$$

where $\mathcal{B}\mathsf{un}^{\tau,0} := *$.

With Notation 3.49 we notice from the Definition 3.46 that the cellular realization functor factors

$$\mathbf{\Theta}_n^{\mathsf{op}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}_{\leq 1},n} \xrightarrow{\circ} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} ~.$$

Lemma 3.50. *The functor $\mathbf{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}_{\leq 1},n}$ is fully faithful, and it carries Segal covers to purely closed covers.*

Proof. Lemma 3.44 gives that $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}_1^{\mathsf{vfr}}$ is fully faithful. Thereafter, Observation 3.31 gives that $\mathbf{\Theta}_n^{\mathsf{op}} \to (\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}})^{\wr n}$ is fully faithful. Lastly, Corollary 3.34 gives that $\mathbf{\Theta}_n^{\mathsf{op}} \to (\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}})^{\wr n} \to \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}_{\leq 1},n}$ is fully faithful. The statement about covers follows immediately by induction, with the case $n = 1$ given by Lemma 3.44.

□

Lemma 3.51. *For each dimension n, the functor*

$$\mathbf{\Theta}_n^{\mathsf{op}} \xrightarrow{\langle - \rangle} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$$

caries Segal covers to purely closed covers.

Proof. Lemma 3.44 states the case for $n = 1$. This implies the assertion for the functor $\mathbf{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}_{\leq 1},n}$. Using Lemma 3.50 just above, it remains to verify that the functor $\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}_{\leq 1},n} \xrightarrow{\circ} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ carries purely closed covers to purely closed covers. Observation 2.40 gives that purely closed covers in $\mathcal{B}\mathsf{un}^{\tau}$ are detected by the projection $\mathcal{B}\mathsf{un}^{\tau} \to \mathcal{B}\mathsf{un}$. So it is enough to verify that the functor $\mathcal{B}\mathsf{un}^{n} \xrightarrow{\circ} \mathcal{B}\mathsf{un}$ carries purely closed covers to purely closed covers. By induction, it is enough to consider the case $n = 2$. This case follows from the following observation, which follows because the sheaf $\mathcal{B}\mathsf{un}$ on Strat is a striation sheaf, and in particualr it is *cone-local* (see §4 of [AFR] for a discussion of this term):

> For each pushout diagram in Strat
>
> $$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & Y'' \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$
>
> in which each morphism is a proper constructible embedding, and for each constructible bundle $X \to Y$, the square of pullbacks
>
> $$\begin{tikzcd} X_{|Y_0} \arrow[r] \arrow[d] & X_{|Y''} \arrow[d] \\ X_{|Y'} \arrow[r] & X \end{tikzcd}$$
>
> too is a pushout.

□

Theorem 3.52. *For each dimension n, the cellular realization functor*

$$\langle - \rangle \colon \boldsymbol{\Theta}_n^{\mathsf{op}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$$

is fully faithful.

Proof. Let $T, T' \in \boldsymbol{\Theta}_n$ be objects. We must show that the map of spaces

$$\boldsymbol{\Theta}_n(T', T) \xrightarrow{\langle - \rangle} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\langle T \rangle, \langle T' \rangle\big) \tag{14}$$

is an equivalence. We will prove (14) is an equivalence by induction on the dimension of the underlying stratified space of $\langle T' \rangle$. Suppose the dimension of $\langle T' \rangle$ is zero. Necessarily, $T' = c_0$ is the 0-cell and $\langle T' \rangle = \mathbb{D}^0$ is the hemispherical 0-disk, which is just $*$. As so, both $\boldsymbol{\Theta}_n(c_0, T)$ and $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\langle T \rangle, \mathbb{D}^0\big)$ are compatibly identified as the space $\langle T \rangle_0$ which is the 0-dimensional stratum of the stratified space $\langle T \rangle$. This proves the base case of our induction.

Now suppose (14) is an equivalence whenever the dimension of the underlying stratified space of $\langle T' \rangle$ is less than k'. We proceed by induction on the dimension of the underlying stratified space of $\langle T \rangle$. Suppose the dimension of $\langle T \rangle$ is zero. Necessarily, $T = c_0$ is the 0-cell and $\langle T \rangle = \mathbb{D}^0$ is the hemispherical 0-disk, which is just $*$. As so, both $\boldsymbol{\Theta}_n(T', c_0)$ and $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\mathbb{D}^0, \langle T' \rangle\big)$ are terminal. This proves the base case of our nested induction. So suppose (14) is an equivalence whenever the dimension of the underlying stratified space of $\langle T \rangle$ is less than k.

By construction, each object of $\boldsymbol{\Theta}_n^{\mathsf{op}}$ can be witnessed as a finite iteration of Segal covers among the cells c_k $(0 \leq k \leq n)$. Because Segal covers are in particular limit diagrams in $\boldsymbol{\Theta}_n^{\mathsf{op}}$, there is a finite limit diagram $\mathcal{U}^{\triangleleft} \to \boldsymbol{\Theta}_n^{\mathsf{op}}$ whose value on the cone point is T' and whose value on each $U \in \mathcal{U}$ is a k-cell for some $0 \leq k \leq n$. Lemma 3.51 gives that the composite functor $\mathcal{U}^{\triangleleft} \to \boldsymbol{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ too is a finite limit diagram whose value on the cone-point is $\langle T' \rangle$ and whose value on each $U \in \mathcal{U}$ is a hemispherical k-disk for some $0 \leq k \leq n$. This explains the vertical equivalences

$$\begin{tikzcd}
\boldsymbol{\Theta}_n(T', T) \arrow[r, "\langle - \rangle"] \arrow[d, "\simeq"'] & \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\langle T \rangle, \langle T' \rangle\big) \arrow[d, "\simeq"] \\
\lim\limits_{U \in \mathcal{U}} \boldsymbol{\Theta}_n(U, T) \arrow[r, "\langle - \rangle"] & \lim\limits_{U \in \mathcal{U}} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\langle T \rangle, \langle U \rangle\big).
\end{tikzcd}$$

Therefore, the map (14) is an equivalence if and only if it is for $T' = c_{j'}$ for each $0 \leq j' \leq n$. '

Lemma 3.51 gives that each object $T \in \boldsymbol{\Theta}_n^{\mathsf{op}}$ can be witnessed as a finite colimit diagram $\mathcal{V}^{\triangleright} \to \boldsymbol{\Theta}_n^{\mathsf{op}}$ with the value on the cone-point T and with the value on each $V \in \mathcal{V}$ a k-cell for some $0 \leq k \leq n$. Lemma 3.53 grants that the composite functor $\mathcal{V}^{\triangleright} \to \boldsymbol{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ is again a finite colimit diagram whose value on the cone-point is $\langle T \rangle$ and whose value on each $V \in \mathcal{V}$ is a hemispherical k-disk for some $0 \leq k \leq n$. This explains the vertical equivalences

$$\begin{tikzcd}
\boldsymbol{\Theta}_n(T', T) \arrow[r, "\langle - \rangle"] \arrow[d, "\simeq"'] & \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\langle T \rangle, \langle T' \rangle\big) \arrow[d, "\simeq"] \\
\lim\limits_{V \in \mathcal{V}} \boldsymbol{\Theta}_n(T', V) \arrow[r, "\langle - \rangle"] & \lim\limits_{V \in \mathcal{V}} \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\langle V \rangle, T'\big).
\end{tikzcd}$$

Therefore, the map (14) is an equivalence if and only if it is for $T = c_j$ for each $0 \leq j \leq n$.

We have reduced the problem of showing (14) is an equivalence to the case $T = c_k$ and $T' = c_{k'}$, with the assumption that the map (14) is an equivalence whenever the dimensions of $\langle T \rangle$ and $\langle T' \rangle$ are smaller than k and k', respectively. Lemma 2.41 states a closed-active factorization system on $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$. Lemma 3.50 implies such a factorization system for $\boldsymbol{\Theta}_n^{\mathsf{op}}$. This is to say that the composition maps from the coends

$$\circ \colon \boldsymbol{\Theta}_n^{\mathsf{act}}(-, T) \bigotimes_{\boldsymbol{\Theta}_n^{\sim}} \boldsymbol{\Theta}_n^{\mathsf{cls}}(T', -) \xrightarrow{\simeq} \boldsymbol{\Theta}_n(T', T) \tag{15}$$

and

$$\circ\colon \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\mathsf{cls}}\big(\langle T\rangle,-\big)\bigotimes_{\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\sim}}\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\mathsf{act}}\big(-,\langle T'\rangle\big)\xrightarrow{\simeq}\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}\big(\langle T\rangle,\langle T'\rangle\big) \tag{16}$$

are equivalences. Now, consider a composable pair of morphisms $\mathbb{D}^k \to D_0 \to \mathbb{D}^{k'}$ in $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ in which the first morphism is closed and the second is active. The target of a closed morphism from $\mathbb{D}^k$ is either $\mathbb{D}^l$ or $\partial\mathbb{D}^l$ for some $k \geq l \leq k'$.

We now rule out the case that this target is $\partial\mathbb{D}^l$ for some $l \leq k$. We show by contradiction that there are no active morphisms from $\partial\mathbb{D}^l$ to $\mathbb{D}^{k'}$. So assume there exists an active morphism $\partial\mathbb{D}^l \xrightarrow{a} \mathbb{D}^{k'}$. Necessarily, $l-1 \leq k'$. Consider the composition $\mathbb{D}^{l-1} \xrightarrow{i} \partial\mathbb{D}^l \xrightarrow{a} \mathbb{D}^{k'}$ with the standard creation morphism of Example 2.50 – this composite is again an active morphism. By induction, this composite morphism $\mathbb{D}^{l-1} \xrightarrow{ai} \mathbb{D}^{k'}$ is in the image of $\mathbf{\Theta}_n^{\mathsf{op}}$. There is a unique such morphism in $\mathbf{\Theta}_n^{\mathsf{op}}$, which is $\mathbb{D}^{l-1} \xrightarrow{i} \mathbb{D}^{k'}$, and it does not factor through $\partial\mathbb{D}^{l-1}$. This is a contradiction.

The conclusion of the previous paragraph is that $D_0 \simeq \mathbb{D}^l$. With this, the equivalences (15) and (16) reduce us, by induction, to showing that each of the two maps

$$\mathbf{\Theta}_n^{\mathsf{cls},\mathsf{op}}(c_k,c_k)\xrightarrow{\langle-\rangle}\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\mathsf{cls}}\big(\mathbb{D}^k,\mathbb{D}^k\big)\qquad\text{and}\qquad\mathbf{\Theta}_n^{\mathsf{act},\mathsf{op}}(c_{k'},c_{k'})\xrightarrow{\langle-\rangle}\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\mathsf{act}}\big(\mathbb{D}^{k'},\mathbb{D}^{k'}\big)$$

is an equivalence of spaces. However, every closed endomorphism $\mathbb{D}^k \to \mathbb{D}^k$ is an isomorphism in $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$. We are thus reduced to showing the righthand equivalence which is just for spaces of active morphisms between cells of the same dimension. We will now explain the solid diagram among mapping spaces

$$\begin{tikzcd}
\mathbf{\Theta}_n^{\mathsf{act},\mathsf{op}}(c_k,c_k) \arrow[rr] \arrow[d, "\langle-\rangle"'] & & \underset{c_l\xrightarrow{\mathsf{cls}}c_k,\,l<k}{\lim}\ \mathbf{\Theta}_n(c_l,c_k) \arrow[d, "\simeq"', "\langle-\rangle"] \\
\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\mathsf{act}}(\mathbb{D}^k,\mathbb{D}^k) \arrow[r] \arrow[d, dashed, "(b)"] \arrow[dr, dashed, "(a)"] & \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}(\mathbb{D}^k,\partial\mathbb{D}^k) \arrow[r, "\simeq", "(1)"'] \arrow[dr, "(2)", "\simeq"'] & \underset{\mathbb{D}^k\xrightarrow{\mathsf{cls}}\mathbb{D}^l,\,l<k}{\lim}\ \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}(\mathbb{D}^k,\mathbb{D}^l) \\
\mathsf{Aut}_{\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}}(\partial\mathbb{D}^k) \arrow[r] & \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\mathsf{act}}(\partial\mathbb{D}^k,\partial\mathbb{D}^k) \arrow[r] & \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}(\partial\mathbb{D}^k,\partial\mathbb{D}^k).
\end{tikzcd}$$

The middle horizontal equivalence (1) is from the standard hemispherical closed cover of the hemispherical $(k-1)$-sphere $\partial\mathbb{D}^k$ by hemispherical disks, each of dimension smaller than k. The upper left vertical arrow is induced by the cellular realization functor; it is an equivalence by induction. The lower diagonal equivalence (2) is from the closed-active factorization system on $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$. The bottom horizontal maps are inclusions of components. We will now argue the existence of the dashed arrow (a)

Consider a constructible bundle $E \to \Delta^1$ classified by an active endomorphism in $\mathcal{B}\mathsf{un}$ of $\mathbb{D}^k$. Consider the fiberwise boundary $\partial E \subset E$. The projection $\partial E \to \Delta^1$ is a constructible bundle classifying an endomorphism in $\mathcal{B}\mathsf{un}$ of $\partial\mathbb{D}^k$. For the morphism in $\mathcal{B}\mathsf{un}$ classifying $E \to \Delta^1$ to be active, it is equivalent to the condition that the continuous map $E \to \Delta^1$ is a fiber bundle of underlying topological spaces. It follows that the constructible bundle $\partial E \to \Delta^1$ too is a fiber bundle of underlying topological spaces. Thereafter, it is classified by an *active* endomorphism of $\partial\mathbb{D}^k$. With this consideration, we conclude the factorization which is (a).

Continuing with the situation of the previous paragraph, should $\partial E \to \Delta^1$ be a fiber bundle of stratified spaces (not just underlying topological spaces), then necessarily $E \to \Delta^1$ too is a conically smooth fiber bundle of stratified spaces. We conclude that the inclusion of components

$$\mathsf{Aut}_{\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}}(\partial\mathbb{D}^k)\xrightarrow{\simeq}\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr},\mathsf{act}}(\mathbb{D}^k,\mathbb{D}^k)$$

is an equivalence of spaces. This provides the factorization which is (b). Because $\mathbf{\Theta}_n^{\mathsf{aut}}(c_k, c_k) \simeq *$ is terminal, we have reduced the problem of showing (14) is an equivalence to that of showing the space of automorphisms

$$\mathsf{Aut}_{\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}}(\partial\mathbb{D}^k) \ \simeq \ *$$

is terminal. This is the statement of Corollary 3.19.

□

The following two lemmas were used in the above proof of Theorem 3.52, the fully faithfulness of cellular realization.

Lemma 3.53. *For each inert morphism $T \to T'$ in $\mathbf{\Theta}_n^{\mathsf{op}}$ there is a natural section $T' \to T$. Furthermore, for each solid pullback diagram in $\mathbf{\Theta}_n^{\mathsf{op}}$ among inert morphisms*

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T'' \arrow[d] \arrow[l, bend right, dashed] \\ T' \arrow[r] \arrow[u, bend left, dashed] & T_0 \arrow[l, bend left, dashed] \arrow[u, bend right, dashed] \end{tikzcd}$$

the natural sections determine a pushout diagram.

Proof. We first consider the $n = 1$ case where $\mathbf{\Theta}_1^{\mathsf{op}} = \mathbf{\Delta}^{\mathsf{op}}$. Let ρ^{op} be an inert morphism in $\mathbf{\Delta}^{\mathsf{op}}$ from $[q]$ to $[p]$; it is the data of a map of linearly ordered sets $[p] \xrightarrow{\rho} [q]$. We define the section $\sigma^{\mathsf{op}} \colon [p] \to [q]$ to the morphism ρ^{op} in $\mathbf{\Delta}^{\mathsf{op}}$ as the data of the map of linearly ordered sets $[q] \xrightarrow{\sigma} [p]$ given as follows. Declare $\sigma(i) = \rho^{-1}(i)$ whenever i lies in the image of ρ; declare $\sigma(i) = 0$ whenever $i < \rho(j)$ for all $j \in [p]$; declare $\sigma(i) = q$ whenever $i > \rho(j)$ for all $j \in [p]$. Using that ρ^{op} is inert, these assignments are well-defined and respect linear orders. The construction of this section is functorial in the following sense: For $T_0 \to T' \to T$ a composition of inert morphisms in $\mathbf{\Delta}^{\mathsf{op}}$, the section of the composite $T_0 \hookrightarrow T$ is the composite of the sections.

It remains to show that if $T = T' \underset{T_0}{\times} T''$ is a pullback in $\mathbf{\Delta}^{\mathsf{op}}$ among inert morphisms then T is the pushout of the diagram formed by the sections. Because the square is comprised of inert morphisms, the pullback in question indeed exists. For the same reason, as a linearly ordered set, the underlying set of this pullback is the pushout of the underlying sets of the constituents of the pullback. The desired pushout follows by inspecting the constructions of the sections to the inert morphisms.

Should $n = 0$, the result is trivially true. We proceed by induction on $n > 0$. Assume the result for $\mathbf{\Theta}_{n-1}^{\mathsf{op}}$. Consider an inert morphism $([p], (T_i)_{0<i\leq p}) \to ([p'], (T_{i'})_{0<i'\leq p'})$ in $\mathbf{\Theta}_n^{\mathsf{op}}$. By definition, this is the data of an inert morphism $[p] \xrightarrow{\rho^{\mathsf{op}}} [p']$ in $\mathbf{\Delta}^{\mathsf{op}}$ together with, for each $0 < i' \in p'$, an inert morphism $T_{\rho(i')} \to T'_{i'}$ in $\mathbf{\Theta}_{n-1}^{\mathsf{op}}$. By induction, there is a natural section $[p'] \to [p]$ as well as a natural section $T'_{i'} \to T_{\rho(i')}$ for each $0 < i' \leq p'$. Together, these define a section $([p'], (T_{i'})_{0<i'\leq p'}) \to ([p], (T_i)_{0<i\leq p})$ in $\mathbf{\Theta}_n^{\mathsf{op}}$. Because it is so for $\mathbf{\Delta}^{\mathsf{op}}$ and $\mathbf{\Theta}_{n-1}^{\mathsf{op}}$ by induction, the construction of this section associated to the given inert morphism is functorial: the composition of the sections is the section of the composite. It remains to check that this process converts an inert colimit diagram in $\mathbf{\Theta}_n$ to a limit diagram. For this, we use that the wreath construction is a right adjoint; specifically, we can recognize a diagram in $\mathbf{\Delta}^{\mathsf{op}} \wr \mathcal{C}$ as a limit if the corresponding diagram in $\mathbf{\Delta}^{\mathsf{op}}$ is a limit and the corresponding diagram in $\mathcal{C}$ is a limit.

□

Lemma 3.54. *The cellular realization $\langle - \rangle : \mathbf{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ preserves the dashed colimits of Lemma 3.53.*

Proof. Each section $\langle T' \rangle \to \langle T \rangle$ is a creation, which is to say that it lies in the image of the limit preserving monomorphism $\mathcal{S}\mathsf{trat}^{\mathsf{cbl,op}} \to \mathcal{B}\mathsf{un}$. As a diagram in $\mathcal{S}\mathsf{trat}^{\mathsf{cbl}}$, it is a pullback. □

4. Factorization homology

We now give a definition of factorization homology. We do this in two conceptual steps. The first step can be interpreted as extending sheaves from a basis for a topology. The second step can be interpreted as integration. We leave to later works a thorough examination of the properties of factorization homology.

4.1. **Higher categories.** We recall a slight modification of Rezk's definition of (∞, n)-categories.

Definition 4.1 (After [Re2])**.** The ∞-category $\mathsf{Cat}_{(\infty,n)}$ of (∞, n)-categories is equipped with a functor $\mathbf{\Theta}_n \to \mathsf{Cat}_{(\infty,n)}$ and is initial among all such for which

- $\mathsf{Cat}_{(\infty,n)}$ is presentable;
- **Segal:** The functor $\mathbf{\Theta}_n \to \mathsf{Cat}_{(\infty,n)}$ carries Segal covering diagrams to colimit diagrams;
- **Univalent:** The functor $\mathbf{\Theta}_n \to \mathsf{Cat}_{(\infty,n)}$ carries univalence diagrams to colimit diagrams.

Necessarily, the given functor $\mathbf{\Theta}_n \to \mathsf{Cat}_{(\infty,n)}$ is fully faithful. As so, the restricted Yoneda functor gives a presentation

$$\mathsf{Cat}_{(\infty,n)} \hookrightarrow \mathsf{PShv}(\mathbf{\Theta}_n)$$

as the full ∞-subcategory consisting of those functors $\mathcal{C}\colon \mathbf{\Theta}_n^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ that carry the opposites of both Segal covering diagrams and univalence diagrams to limit diagrams. We will sometimes refer to such presheaves as *univalent Segal* $\mathbf{\Theta}_n$*-spaces*.

Example 4.2. For each $n > 0$ consider the ordinary category $n\mathsf{Cat}$ of ordinary categories enriched over the ordinary Cartesian category $(n-1)\mathsf{Cat}$, where $0\mathsf{Cat} := \mathsf{Set}$. In general, for V an ordinary category that admits finite products, there is a functor $\mathbf{\Delta}^{\mathsf{op}} \wr \mathsf{V}^{\mathsf{op}} \to \mathsf{Cat}(\mathsf{V})^{\mathsf{op}}$ to V-enriched categories; this is constructed in [Be]. By induction on n, there results a functor $\mathbf{\Theta}_n \hookrightarrow n\mathsf{Cat}$; in [Be] this functor is shown to be fully faithful. The left Kan extension of the defining functor $\mathbf{\Theta}_n \to \mathsf{Cat}_{(\infty,n)}$ along this functor defines a functor between ∞-categories

$$n\mathsf{Cat} \longrightarrow \mathsf{Cat}_{(\infty,n)} ~.$$

In particular, each strict n-category determines an (∞, n)-category.

Remark 4.3. The work of Barwick and Schommer-Pries [BS] establishes an axiomatic approach to (∞, n)-categories, in the background of quasi-categories. They show that each candidate quasi-category of (∞, n)-categories is equivalent to Rezk's Definition 4.1 above.

Example 4.4. For $n = 1$, the standard functor $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Assoc}$ over Fin_* to the associative ∞-operad determines a fully faithful functor

$$\mathfrak{B}\colon \mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{S}\mathsf{paces}^{\times}) ~\hookrightarrow~ \mathsf{Cat}_{(\infty,1)}^{*/} ~\subset~ \mathsf{PShv}(\mathbf{\Delta})^{*/}$$

from the ∞-category of associative algebras in the Cartesian symmetric monoidal ∞-category of spaces to pointed $(\infty, 1)$-categories. So asscoiative monoids in spaces give examples of $(\infty, 1)$-categories.
(The fact that this functor $\mathfrak{B}$ factors as indicated is because $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Assoc}$ is an *approximation*, as developed in §2.3.3 of [Lu2].)

Example 4.5. For $n > 0$, the previous Example 4.4 inductively defines a functor

$$\mathbf{\Theta}_n^{\mathsf{op}} \simeq \mathbf{\Delta}^{\mathsf{op}} \wr \mathbf{\Theta}_{n-1}^{\mathsf{op}} \longrightarrow \mathcal{E}_1 \wr \mathcal{E}_{n-1} \xrightarrow{\times} \mathcal{E}_n$$

over Fin_* to the ∞-operad of little n-disks. This functor determines a functor

$$\mathfrak{B}^n\colon \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{S}\mathsf{paces}^{\times}) \longrightarrow \mathsf{Cat}_{(\infty,n)}^{*/} ~\subset~ \mathsf{PShv}(\mathbf{\Theta}_n)^{*/}$$

from $\mathcal{E}_n$-algebras in the Cartesian symmetric monoidal ∞-category of spaces to pointed (∞, n)-categories. So $\mathcal{E}_n$-algebras in spaces give examples of (∞, n)-categories. In particular, $\mathcal{E}_\infty$-algebras in spaces, such as commutative monoids and infinite-loop spaces, give examples of (∞, n)-categories. (The fact that this functor $\mathfrak{B}^n$ factors as indicated is because $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Assoc}$ is an *approximation*,

because the wreath construction respects inert-coCartesian morphisms (see §2.4.4 of [Lu2]), and because the map from the wreath to $\mathcal{E}_n$ respects inert-coCartesian morphsims (see §5.1.2 of [Lu2]).)

Remark 4.6. Example 4.5 in particular gives that an $\mathcal{E}_\infty$-space determines an (∞,n)-category for each $n\geq 0$. This can be achieved more directly through the foundational work of Segal in [Se1]. Namely, in that work Segal shows how an infinite-loop space C determines a functor $C\colon \mathsf{Fin}_*\to \mathcal{S}\mathsf{paces}$ that satisfies a *reduced* condition, meaning $C(*)\simeq *$ is terminal, and what has been termed the *Segal* condition, meaning C carries each pullback diagram in Fin_* among *inert* morphisms to pullback squares among spaces. (In this situation of based finite sets, a based map $I_+\to J_+$ is *inert* if the restriction $I_{|J}\to J$ is an isomorphism.)

Example 4.7. For each $0\leq k\leq n$, right Kan extension along the functor $\iota_k\colon \mathbf{\Theta}_k\hookrightarrow\mathbf{\Theta}_n$ of Observation 3.43 defines a functor between ∞-categories

$$(\iota_k)_*\colon \mathsf{Cat}_{(\infty,k)}\ \hookrightarrow\ \mathsf{Cat}_{(\infty,n)}$$

which is fully faithful. So (∞,k)-categories give examples of (∞,n)-categories. In particular, there is a fully faithful functor from spaces, regarded as ∞-groupoids:

$$\mathcal{S}\mathsf{paces}\simeq\mathsf{Cat}_{(\infty,0)}\ \hookrightarrow\ \mathsf{Cat}_{(\infty,n)}~.$$

4.2. **Labeling systems from higher categories.** Toward the construction of factorization homology from higher categories, we explain here how an (∞,n)-category determines a labeling system on a sufficiently finely stratified vari-framed n-manifold.

The next result makes use of the cellular realization functor $\langle-\rangle\colon \mathbf{\Theta}_n^{\mathsf{op}}\hookrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ of Definition 3.46, which Theorem 3.52 verifies is fully faithful. We postpone the proof of this result to the end of this section.

Lemma 4.8. *The restricted Yoneda functor* $(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}})^{\mathsf{op}}\to\mathsf{PShv}(\mathbf{\Theta}_n)$ *takes values in* (∞,n)*-categories.*

Notation 4.9. After Lemma 4.8, we denote the factorized restricted Yoneda functor as

$$\mathfrak{C}\colon (\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}})^{\mathsf{op}}\longrightarrow\mathsf{Cat}_{(\infty,n)}~.$$

Remark 4.10. The construction of $\mathfrak{C}$, as a functor among ∞-categories, embodies many of the choices and constructions in this article. To given some intuition for the value $\mathfrak{C}(M)$ on a compact vari-framed disk-stratified n-manifold, there is an non-identity i-morphism of this (∞,n)-category for each connected i-dimensional strata of M.

Consider the sequence of functors

$$\mathsf{Cat}_{(\infty,n)}\longrightarrow\mathsf{PShv}(\mathbf{\Theta}_n)\longrightarrow\mathsf{Fun}(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}},\mathcal{S}\mathsf{paces})\tag{17}$$

in which the second is given by right Kan extension. Explicitly, for $\mathcal{C}$ an (∞,n)-category, this right Kan extension evaluates on a compact vari-framed disk-stratified n-manifold M as the space

$$\mathsf{Cat}_{(\infty,n)}\big(\mathfrak{C}(M),\mathcal{C}\big)$$

of functors between (∞,n)-categories.

Remark 4.11. We think of the space $\mathsf{Cat}_{(\infty,n)}\big(\mathfrak{C}(M),\mathcal{C}\big)$ appearing in the above expression as the space of $\mathcal{C}$-labeling systems on the vari-framed disk-stratified n-manifold M. In other words, the (∞,n)-category corepresents M-labeling systems.

Corollary 4.12. *The composite functor (17)*

$$\mathsf{Cat}_{(\infty,n)}\ \hookrightarrow\ \mathsf{Fun}(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}},\mathcal{S}\mathsf{paces})~,\qquad \mathcal{C}\mapsto\Big(M\mapsto\mathsf{Cat}_{(\infty,n)}\big(\mathfrak{C}(M),\mathcal{C}\big)\Big)$$

is fully faithful.

Proof. The functor is given as a composition of two functors. The first is fully faithful, from the defining universal property of the ∞-category $\mathsf{Cat}_{(\infty,n)}$, as discussed above. That the second functor is fully faithful is an immediate because the cellular realization $\langle - \rangle\colon \mathbf{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ is fully faithful (Theorem 3.52).

□

Proof of Lemma 4.8. We will consider the restricted Yoneda functor $\mathfrak{C}\colon (\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}})^{\mathsf{op}} \to \mathsf{PShv}(\mathbf{\Theta}_n^{\mathsf{op}})$ which enlarges the restricted Yoneda functor of Notation 4.9. Lemma 3.51 directly states that Segal covering diagrams are carried by the cellular realization functor to purely closed covers. Because purely closed covers are in particular limit diagrams in $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$, each presheaf $\mathfrak{C}(M)$ on $\mathbf{\Theta}_n$ carries the opposites of Segal covering diagrams to limit diagrams of spaces.

It remains to verify that each value $\mathfrak{C}(M)$ carries the opposites of univalence diagrams to limit diagrams of spaces. For this we use use the criterion of Lemma 5.22, through which it is enough to show that the only k-idempotents of the Segal $\mathbf{\Theta}_n$-space $\mathfrak{C}(M)$ are identity k-morphisms.

Consider the unique (up to equivalence) vari-framed 1-manifold D whose underlying stratified space is the pushout $\ast \underset{\partial\mathbb{D}^1}{\amalg} \mathbb{D}^1$ in Strat. Up to equivalence, there is a unique refinement morphism $D \to S^1$. As so, for each $p \geq 0$, there is a fiber bundle $\widetilde{D}_p \to D$ among stratified spaces which refines the p-sheeted covering map $S^1 \xrightarrow{z \mapsto z^p} S^1$, as well as a refinement $\widetilde{D}_p \to D$, of which there are p. There results a composite morphism $F_p\colon D \xrightarrow{\mathsf{cr}} \widetilde{D}_p \xrightarrow{\mathsf{ref}} D$ in $\mathsf{c}\mathcal{M}\mathsf{fd}_1^{\mathsf{vfr}}$. Thereafter, for each $0 < k \leq n$, there is a morphism $F_p^k\colon D^k \to D^k$ in $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ among $(k-1)$-fold framed suspensions, $D^k := \mathsf{S}^{\mathsf{fr},\circ(k-1)}(D)$.

From the definition of the stratified space D as a pushout in Strat, there is a canonical map of stratified spaces $\mathbb{D}^1 \xrightarrow{u} D$. This map is a constructible bundle, and therefore determines the morphism $D \to \mathbb{D}^1$ in $\mathsf{c}\mathcal{B}\mathsf{un}$ which classifies the reversed mapping cylinder of the map of stratified spaces: $\mathsf{Cylr}(u) \to \Delta^1$. This cylinder is equipped with a vertical vari-framing: $\mathsf{Exit}(\mathsf{Cylr}(u)) \to \mathsf{vfr}$ over $\mathcal{E}\mathsf{xit}$. We arrive at a creation morphism $D \to \mathbb{D}^1$ in $\mathsf{c}\mathcal{M}\mathsf{fd}_1^{\mathsf{vfr}}$. Taking iterated framed suspensions determines a creation morphism

$$D^k := \mathsf{S}^{\mathsf{fr},\circ(k-1)}(D) \longrightarrow \mathsf{S}^{\mathsf{fr},\circ(n-1)}(\mathbb{D}^1) \cong \mathbb{D}^k$$

in $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ for each $0 < k \leq n$. Applying $\mathfrak{C}$ determines a map $c_k \to \mathfrak{C}(D^k)$ of presheaves on $\mathbf{\Theta}_n$. By inspection, this map factors through an equivalence from a quotient of a k-cell

$$c_k \simeq c_{k-1} \wr c_1 \longrightarrow c_{k-1} \wr (c_1/\partial c_1) \;\simeq\; \mathfrak{C}(D^k)$$

of presheaves on $\mathbf{\Theta}_n^{\mathsf{op}}$. As so the Segal space $\mathfrak{C}(D^k)$ is univalent, and therefore presents an (∞,n)-category; this (∞,n)-category $\mathfrak{C}(D^k)$ corepresents a k-endomorphism of a $(k-1)$-morphism of an (∞,n)-category. As so, there is a preferred functor $\mathfrak{C}(D^k) \xrightarrow{e^k} \mathsf{Idem}^k$ between (∞,n)-categories, together with, for each $p \geq 0$, an identification of functors $e^k \circ \mathfrak{C}(F_p^k) \simeq e^k$.

So, to prove that each functor $\mathsf{Idem}^k \to \mathfrak{C}(M)$ factors through $c_{k-1} \to \mathfrak{C}(M)$ it enough to prove that each morphism $M \xrightarrow{\epsilon^k} D^k$ in $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$ factors through the standard creation map $\mathbb{D}^{k-1} \to D^k$ whenever there is an identification of morphisms $\epsilon^k \circ F_p^k \simeq \epsilon^k$ in $\mathsf{c}\mathcal{M}\mathsf{fld}_n^{\mathsf{vfr}}$. This statement is implied by its likewise version without vari-framings. Namely, it is enough to prove the following assertion.

For each commutative diagram in $\mathsf{c}\mathcal{B}\mathsf{un}$

$$\begin{tikzcd} M \arrow[rr, "\epsilon^k"] \arrow[dr, "\epsilon^k"'] & & D^k \arrow[dl, "F_2^k"] \\ & D^k & \end{tikzcd}$$

there is a factorization $\epsilon^k\colon M \xrightarrow{\mathsf{cls}} M_{<k} \to D^k$ through the unit of the adjunction of Lemma 3.27 which is given by forgetting strata of dimension at least k.

Using the closed-active factorization system on $\mathsf{c}\mathcal{B}\mathsf{un}$ (Lemma 1.20), it is enough to consider the case that the morphism ϵ^k is active. In this case the assertion becomes the statement that the topological dimension of M is less than k.

Such a commutative diagram in $\mathsf{c}\mathcal{B}\mathsf{un}$ is a constructible bundle $X \to \Delta^2$ together with identifications of its restrictions along $\Delta^S \subset \Delta^2$ for various non-empty linearly ordered subsets $\emptyset \neq S \subset \{0 < 1 < 2\}$. Consider the link system among stratified spaces

$$M \xleftarrow[\mathsf{p.cbl.surj}]{\pi} \mathsf{Link}_M(X) \xrightarrow[\mathsf{ref}]{\gamma} X_{|\Delta^{\{1<2\}}} ~;$$

here, the leftward map is proper and constructible and surjective while the rightward map is a refinement. The naturality of links grants a surjective proper constructible bundle $\mathsf{Link}_M(X) \to \mathsf{Link}_{\Delta^{\{0\}}}(\Delta^2) \cong \Delta^1$. We obtain a surjective proper constructible bundle $\mathsf{Link}_M(X) \to M \times \Delta^1$. Because the topological dimension of the stratified space D^k is bounded above by k, then the topological dimension of the $\mathsf{Link}_M(X)$ is bounded above by $(k+1)$, and therefore the topological dimension of M is bounded above by k.

Let $M_k \subset M$ be the open subspace consisting of the k-dimensional strata; we must explain why M_k is empty. For dimension reasons, the surjective proper constructible bundle $\mathsf{Link}_M(X)_{|M_k \times \Delta^1} \to M_k \times \Delta^1$ has finite fibers which are empty if and only if M_k is empty. By assumption, there is an identification of stratified spaces $\mathsf{Link}_M(X)_{|M_k \times \Delta^{\{0\}}} \cong \mathsf{Link}_M(X)_{|M_k \times \Delta^{\{1\}}}$ over M_k. On the other hand, from the construction of the morphism F_2^k in $\mathsf{c}\mathcal{B}\mathsf{un}$, the cardinality of the fiber $\mathsf{Link}_M(X)_{|\{x\} \times \Delta^{\{0\}}}$ is twice that of the fiber $\mathsf{Link}_M(X)_{|\{x\} \times \Delta^{\{1\}}}$ for each $x \in M_k$. We conclude that $M_k = \emptyset$, as desired. $\square$

4.3. **Factorization homology.** As Corollary 4.12 we identified (∞, n)-categories as a full ∞-subcategory of copresheaves on compact disk-stratified vari-framed n-manifolds. By definition, such stratified disks lie fully among all such stratified manifolds. We can now define factorization homology as the left Kan extension from this full ∞-subcategory to copresheaves of manifolds.

Definition 4.13. Factorization homology is the composite

$$\int : \mathsf{Cat}_{(\infty,n)} \hookrightarrow \mathsf{Fun}\big(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces}\big) \hookrightarrow \mathsf{Fun}\big(\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces}\big)$$

of right Kan extension along $\mathbf{\Theta}_n^{\mathsf{op}} \hookrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ followed by left Kan extension along $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \hookrightarrow \mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$.

Equivalently, given an (∞, n)-category $\mathcal{C}$, factorization homology is defined by the following two Kan extensions:

$$\begin{tikzcd}
\mathbf{\Theta}_n^{\mathsf{op}} \arrow[r, "\mathcal{C}"] \arrow[d, hook] & \mathcal{S}\mathsf{paces} \\
\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \arrow[ur, dashed, "\mathcal{C}"'] \arrow[d, hook] & \\
\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}} \arrow[uur, dashed, "\int \mathcal{C}"'] &
\end{tikzcd}$$

This left Kan extension is equivalent to the classifying space of the unstraightening construction $\mathcal{C}_M$ of the composite

$$\mathsf{c}\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{vfr}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \xrightarrow{\mathcal{C}} \mathcal{S}\mathsf{paces} ~.$$

This unstraightening construction can be thought of the ∞-category of $\mathcal{C}$-labeled disk-stratifications over M. By that token we can describe the factorization homology

$$\int_M \mathcal{C} ~\simeq~ \mathsf{B}\big(\mathcal{C}_M\big)$$

as the classifying space of $\mathcal{C}$-labeled disk-stratifications over M.

Remark 4.14. For M a compact framed smooth n-manifold, and for $\mathcal{C}$ an (∞, n)-category, heuristically the factorization homology $\int_M \mathcal{C}$ is the integral, or average, of sufficiently fine $\mathcal{C}$-labeled vari-framed refinements of M. This description as a left Kan extension makes this heuristic precise as well as functorial in M up to coherent homotopy.

Examination of this factorization homology, and its development in the enriched case, will take place in other works. Here are some familiar values of this factorization homology.

Example 4.15.

- For each $0 \leq i \leq n$, and for each (∞, n)-category $\mathcal{C}$, the value $\int_{\mathbb{D}^i} \mathcal{C}$ is the space of i-morphisms of $\mathcal{C}$. In particular, $\int_* \mathcal{C} \simeq \mathcal{C}^\sim$.
- For E a connective spectrum, Remark 4.6 provides an (∞, n)-category $\mathfrak{B}^n\Omega^\infty E$, which is equipped with a functor from $*$ that we neglect. For M a compact framed smooth n-manifold, Dold–Thom theory provides the identification of graded abelian groups from the homotopy groups of the factorization homology
$$\pi_*\Big(\int_M \mathfrak{B}^n\Omega^\infty E\Big) \;\cong\; \mathsf{H}_*(M; E)$$
with the homology groups of the underlying space of M with coefficients in the spectrum E.
- Through Example 4.7, we regard each space Z as an (∞, n)-category for any $n \geq 0$. For M a compact framed smooth n-manifold, the factorization homology
$$\int_M Z \;\simeq\; \mathsf{Map}(M, Z)$$
is identified as the space of maps to Z from the underlying space of M.
- For $\mathcal{C}$ an $(\infty, 1)$-category, the value
$$\int_{S^1} \mathcal{C} \;\simeq\; |\mathsf{N}^{\mathsf{cy}}\mathcal{C}|$$
is equivalent to the geometric realization of the cyclic nerve of $\mathcal{C}$.

Remark 4.16. Fix an (∞, n)-category $\mathcal{C}$. The restriction of $\int \mathcal{C} \colon \mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}} \to \mathcal{S}\mathsf{paces}$ to the ∞-subcategory $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr,cr}}$ of creation morphisms reveals some interesting transfer-type behavior of factorization homology. We demonstrate.

- For each compact framed smooth n-manifold M, there is an action of the group object in $\mathcal{S}\mathsf{paces}$ of framed diffeomorphisms of M on factorization homology:
$$\mathsf{Diff}^{\mathsf{fr}}(M) \longrightarrow \mathsf{Aut}\Big(\int_M \mathcal{C}\Big) .$$
In the case that $M = \mathbb{T}^n$ is an n-torus, we see a homotopy coherent action of the Lie group $\mathbb{T}^n$ on the n-fold Hochschild space $\int_{\mathbb{T}^n} \mathcal{C}$ thereby resembling an n-fold Vershibung.
- For each smooth fiber bundle $E \to M$ between compact manifolds whose dimensions are bounded above by n, together with a framing of M and a trivialization of the vertical tangent bundle of this fiber bundle, there is a naturally associated map of spaces
$$\int_M \mathcal{C} \longrightarrow \int_E \mathcal{C} .$$
In the case that $(E \to M) = (S^1 \xrightarrow{z \mapsto z^p} S^1)$ is the standard connected p-sheeted cover of the circle, we see resemblance with the Frobenius maps of the cyclotomic structure anticipated on the Hochschild space $\int_{S^1} \mathcal{C}$. In the case that $(E \to M) = (S^1 \to *)$, this is a map $\mathcal{C}^\sim \to \int_{S^1} \mathcal{C}$ from the underlying ∞-groupoid of $\mathcal{C}$ to its Hochschild space, and we see resemblance of the trace map from algebraic K-theory, as it is invariant with respect to both the Frobenius and Vershibung maps.

5. Appendix: some ∞-category theory

We go over some notions within ∞-category theory.

5.1. **Monomorphisms.** In this section we characterize monomorphisms among ∞-categories. We first recall the following standard definition.

Definition 5.1 (Mono/Epi)**.** A map $X \to Y$ between spaces is a *monomorphism* if, when regarded as a functor between ∞-groupoids, it is fully faithful. A morphism $f\colon [1] \to \mathcal{X}$ in an ∞-category is a *monomorphism* if, for each object $* \xrightarrow{x} \mathcal{X}$, the composite functor

$$[1] \xrightarrow{f} \mathcal{X} \longrightarrow \mathsf{PShv}(\mathcal{X}) \xrightarrow{x^*} \mathsf{PShv}(*) \simeq \mathcal{S}\mathsf{paces}$$

is a monomorphism between spaces. A morphism $f\colon [1] \to \mathcal{X}$ in an ∞-category is an *epimorphism* if $[1] \simeq [1]^{\mathsf{op}} \xrightarrow{f^{\mathsf{op}}} \mathcal{X}^{\mathsf{op}}$ is a monomorphism.

Remark 5.2. Alternatively, a map between spaces $f\colon X \to Y$ is a monomorphism if and only if it is an inclusion of path components. This is to say that, for any choice of base point $x \in X$, the homomorphism between homotopy groups $\pi_q(X; x) \to \pi_q(Y; f(x))$ is an isomorphism for each $q > 0$.

Example 5.3. Let X be a space for which its suspension

$$S(X) := * \coprod_X * \ \simeq \ *$$

is contractible. Because epimorphisms are preserved by co-base change, we conclude that the unique map $X \to *$ is an epimorphism.

Because it is the case for monomorphisms among spaces, both monomorphisms and epimorphisms satisfy a certain two-out-of-three property:

Observation 5.4. Let $[2] \to \mathcal{X}$ be a functor between ∞-categories. For each $0 \leq i < j \leq 2$, denote the restriction $f_{ij}\colon \{i < j\} \to \mathcal{X}$.

- Should f_{01} and f_{12} be monomorphisms, then so is f_{02}.
- Should f_{01} and f_{12} be epimorphisms, then so is f_{02}.
- Should f_{12} and f_{02} be monomorphisms, then so is f_{01}.
- Should f_{01} and f_{02} be epimorphisms, then so is f_{12}.

Monomorphisms, respectively epimorphisms, are closed under the formation of limits, respectively colimits, in the following sense.

Observation 5.5. For $\mathcal{X}$ an ∞-category, the monomorphisms and the epimorphisms form full ∞-subcategories $\mathsf{Ar}^{\mathsf{mono}}(\mathcal{X}) \subset \mathsf{Ar}(\mathcal{X}) \supset \mathsf{Ar}^{\mathsf{epi}}(\mathcal{X})$ of the ∞-category of arrows, $\mathsf{Ar}(\mathcal{X}) := \mathsf{Fun}([1], \mathcal{X})$. Furthermore, each diagram among ∞-categories

$$\begin{tikzcd} \mathcal{J} \arrow[r] \arrow[d] & \mathsf{Ar}^{\mathsf{mono}}(\mathcal{X}) \arrow[d] \\ \mathcal{J}^{\triangleleft} \arrow[r, "\lim"] \arrow[ur, dashed] & \mathsf{Ar}(\mathcal{X}), \end{tikzcd}$$

in which the bottom horizontal arrow is a limit diagram, factors as a limit diagram. Likewise, each diagram among ∞-categories

$$\begin{tikzcd} \mathcal{J} \arrow[r] \arrow[d] & \mathsf{Ar}^{\mathsf{epi}}(\mathcal{X}) \arrow[d] \\ \mathcal{J}^{\triangleright} \arrow[r, "\mathsf{colim}"] \arrow[ur, dashed] & \mathsf{Ar}(\mathcal{X}), \end{tikzcd}$$

in which the bottom horizontal arrow is a colimit diagram, factors as a colimit diagram.

Lemma 5.6. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a monomorphism if and only if the map between maximal ∞-subgroupoids $\mathcal{C}^\sim \to \mathcal{D}^\sim$ is a monomorphism, and, for each pair of objects $\partial[1] \xrightarrow{x \sqcup y} \mathcal{C}$, the map between spaces of morphisms*

$$\mathcal{C}(x,y) \simeq \mathsf{Map}^{x\sqcup y/}([1],\mathcal{C}) \xrightarrow{F\circ -} \mathsf{Map}^{F\circ(x\sqcup y)/}([1],\mathcal{D}) \simeq \mathcal{D}(F(x),F(y))$$

is a monomorphism.

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Restriction along the functor $\partial[1] \to [1]$ determines the downward maps in the diagram of spaces of functors

$$\begin{tikzcd}
\mathsf{Map}_{\mathsf{Cat}_\infty}([1],\mathcal{C}) \arrow[r, "F\circ -"] \arrow[d] & \mathsf{Map}_{\mathsf{Cat}_\infty}([1],\mathcal{D}) \arrow[d] \\
\mathsf{Map}_{\mathsf{Cat}_\infty}(\partial[1],\mathcal{C}) \arrow[r, "F\circ -"] & \mathsf{Map}_{\mathsf{Cat}_\infty}(\partial[1],\mathcal{D}).
\end{tikzcd}$$

Suppose F is a monomorphism. Then the horizontal maps in this diagram are monomorphisms. It follows that, for each pair of objects $x \sqcup y\colon \partial[1] \to \mathcal{C}$, the map of fibers $\mathcal{C}(x,y) \to \mathcal{D}(F(x),F(y))$ is a monomorphism. Also, F being a monomorphism implies the map between spaces $\mathcal{C}^\sim \simeq \mathsf{Map}_{\mathsf{Cat}_\infty}(*,\mathcal{C}) \xrightarrow{F\circ -} \mathsf{Map}_{\mathsf{Cat}_\infty}(*,\mathcal{D}) \simeq \mathcal{D}^\sim$ is a monomorphism.

Now suppose, that the map of spaces $\mathcal{C}^\sim \to \mathcal{D}^\sim$ is a monomorphism and, for each pair of objects $x \sqcup y\colon \partial[1] \to \mathcal{C}$, that the map of spaces $\mathcal{C}(x,y) \to \mathcal{D}\big(F(x),F(y)\big)$ is a monomorphism. Let $\mathcal{K}$ be an ∞-category. We must show the map between spaces of functors

$$\mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{K},\mathcal{C}) \xrightarrow{F\circ -} \mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{K},\mathcal{D})$$

is a monomorphism. First note that the fully faithful functor $\mathbf{\Delta}_{\leq 1} \hookrightarrow \mathsf{Cat}_\infty$ generates Cat_∞ under colimits. That is, for each full ∞-subcategory $\mathbf{\Delta}_{\leq 1} \subset \mathcal{S} \subset \mathsf{Cat}_\infty$ that is closed under the formation of colimits, then the inclusion $\mathcal{S} \to \mathsf{Cat}_\infty$ is an equivalence. Using this, choose a colimit diagram $\mathcal{K}_\bullet\colon \mathcal{J}^\triangleright \to \mathsf{Cat}_\infty$ that carries the cone point to $\mathcal{K}$ and each $j \in \mathcal{J}$ to $[0]$ or $[1]$. Using that monomorphisms are closed under the formation of limits (Observation 5.5), we conclude that the map of spaces of functors

$$\mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{K},\mathcal{C}) \xrightarrow{\simeq} \lim_{j\in\mathcal{J}} \mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{K}_j,\mathcal{C}) \longrightarrow \lim_{j\in\mathcal{J}} \mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{K}_j,\mathcal{D}) \xleftarrow{\simeq} \mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{K},\mathcal{D})$$

is a monomorphism provided it is in the cases that $\mathcal{K} = *$ and $\mathcal{K} = [1]$. The case of $\mathcal{K} = *$ is the assumption that $\mathcal{C}^\sim \to \mathcal{D}^\sim$ is a monomorphism. With the above square diagram, the case of $\mathcal{K} = [1]$ follows from the case of $\mathcal{K} = *$ together with the additional assumption about mapping spaces. □

Example 5.7. Fully faithful functors among ∞-categories are monomorphisms.

Example 5.8. For $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ a functor between ∞-categories, the collection of π-Cartesian morphisms determines a monomorphism $\mathcal{E}^{\mathsf{Cart}/\pi} \hookrightarrow \mathcal{E}$: a functor $\mathcal{K} \to \mathcal{E}$ factors through $\mathcal{E}^{\mathsf{Cart}/\pi}$ whenever each composition $[1] \to \mathcal{K} \to \mathcal{E}$ is a π-Cartesian morphism. In the case that $\mathcal{B} \simeq *$ is terminal, we see that the functor $\mathcal{E}^\sim \hookrightarrow \mathcal{E}$ from the maximal ∞-subgroupoid is a monomorphism.

Example 5.9. For each monomorphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ among ∞-categories, and for each ∞-category $\mathcal{K}$, the functor between functor ∞-categories

$$\mathsf{Fun}(\mathcal{K},\mathcal{C}) \xrightarrow{F\circ -} \mathsf{Fun}(\mathcal{K},\mathcal{D})$$

too is a monomorphism.

5.2. **Cospans.** We record some facts about ∞-categories of cospans. See [Ba] for a more thorough development.

We denote the functor from finite sets to posets,

$$\mathcal{P}_{\neq\emptyset}(-)\colon \mathsf{Fin} \longrightarrow \mathsf{Poset} ~,$$

whose value on a set S is the poset of non-empty subsets of S, ordered by inclusion. We will use the same notation for its precomposition with the forgetful functor $\mathbf{\Delta} \to \mathsf{Fin}$ given by forgetting linear orders.

Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{C}^\sim \subset \mathcal{L}, \mathcal{R} \subset \mathcal{C}$ be a pair of ∞-subcategories each of which contains the maximal ∞-subgroupoid of $\mathcal{C}$. The simplicial space $\mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}}$ is the subfunctor of the composite functor

$$\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{P}_{\neq\emptyset}(-)} \mathsf{Poset}^{\mathsf{op}} \xrightarrow{\mathsf{Map}(-,\mathcal{C})} \mathcal{S}\mathsf{paces}$$

consisting of those values $\mathcal{P}_{\neq\emptyset}(\{0,\dots,p\}) \xrightarrow{F} \mathcal{C}$ that satisfy the following conditions.

- The functor F carries colimit diagrams to colimit diagrams.
- The functor F carries inclusions $S \subset T$ to morphisms in $\mathcal{L}$ whenever $\mathsf{Min}(S) = \mathsf{Min}(T)$.
- The functor F carries inclusions $S \subset T$ to morphisms in $\mathcal{R}$ whenever $\mathsf{Max}(S) = \mathsf{Max}(T)$.

Explicitly, the value $\mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}}[0] \simeq \mathcal{C}^\sim$ is the maximal ∞-subgroupoid of $\mathcal{C}$; the value on $[1]$ is the pullback among spaces

$$\begin{tikzcd}
\mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}}[1] \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{L})^\sim \arrow[d, "\mathsf{ev}_t"] \\
\mathsf{Ar}(\mathcal{R})^\sim \arrow[r, "\mathsf{ev}_t"] & \mathcal{C}^\sim
\end{tikzcd}$$

involving the spaces of arrows in $\mathcal{L}$ and in $\mathcal{R}$ and evaluations at their targets. So a point in $\mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}}$ is an object of $\mathcal{C}$, and a 1-simplex in $\mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}}$ is a *cospan* in $\mathcal{C}$ by $\mathcal{L}$ and $\mathcal{R}$, by which we mean a diagram in $\mathcal{C}$

$$c_- \xrightarrow{l} c_0 \xleftarrow{r} c_+$$

in which l is a morphism in $\mathcal{L}$ and r is a morphism in $\mathcal{R}$.

Criterion 5.10. Let $\mathcal{L} \to \mathcal{C} \leftarrow \mathcal{R}$ be essentially surjective monomorphisms among ∞-categories. Suppose each diagram $c_+ \xleftarrow{r} c_0 \xrightarrow{l} c_-$ in $\mathcal{C}$ admits a pushout whenever the morphism l belongs to $\mathcal{L}$ and r belongs to $\mathcal{R}$. Then the simplicial space $\mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}}$ presents an ∞-category.

We consider an ∞-subcategory

$$\mathsf{Pair} ~\subset~ \mathsf{Fun}\big(\mathcal{P}_{\neq\emptyset}(\{\pm\}), \mathsf{Cat}_\infty\big) ~.$$

The objects are those $\mathcal{L} \to \mathcal{C} \leftarrow \mathcal{R}$ for which both functors are essentially surjective monomorphisms and which satisfy the condition of Criterion 5.10. The morphisms are those which preserve the pushouts of Criterion 5.10. Manifest from its construction, these simplicial spaces of cospans organize as a functor

$$\mathsf{cSpan}\colon \mathsf{Pair} \longrightarrow \mathsf{Cat}_\infty ~, \qquad (\mathcal{C}^\sim \subset \mathcal{L}, \mathcal{R} \subset \mathcal{C}) \mapsto \mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}} ~. \tag{18}$$

By inspection, this functor (18) preserves finite products. The previous criterion thus gives the next observation.

Observation 5.11. The functor (18) lifts as a functor to symmetric monoidal ∞-categories

$$\mathsf{cSpan}\colon \mathsf{Alg}_{\mathsf{Com}}(\mathsf{Pair}^\times) \longrightarrow \mathsf{Cat}_\infty^\otimes$$

from commutative algebras in the Cartesian ∞-operad associated to the ∞-category Pair.

Observation 5.12. For each symmetric monoidal ∞-category $\mathcal{C}$ there is a canonical identification of symmetric monoidal ∞-groupoids

$$\mathcal{C}^{\sim} \ \simeq\ \mathsf{cSpan}(\mathcal{C})^{\mathcal{C}^{\sim}\text{-}\mathcal{C}^{\sim}} \ .$$

In particular, for each symmetric monoidal pair $(\mathcal{L} \subset \mathcal{C} \supset \mathcal{R}) \in \mathsf{Alg}_{\mathsf{Com}}(\mathsf{Pair}^{\times})$, there is a canonical symmetric monoidal functor

$$\mathcal{C}^{\sim} \longrightarrow \mathsf{cSpan}(\mathcal{C})^{\mathcal{L}\text{-}\mathcal{R}}$$

from the maximal symmetric monoidal ∞-subgroupoid.

5.3. **Exponentiability in Cat_∞.** In this section we verify a couple relevant examples of *exponentiable fibrations* $\mathcal{E} \to \mathcal{B}$. See [AF2] for a more extensive treatment of this subject.

5.3.1. **Basic notions.** Each functor $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ between ∞-categories determines a functor

$$\pi_! \colon \mathsf{Cat}_{\infty/\mathcal{E}} \xrightarrow{-\circ\pi} \mathsf{Cat}_{\infty/\mathcal{B}} \ , \qquad (\mathcal{K} \to \mathcal{E}) \mapsto (\mathcal{K} \to \mathcal{E} \xrightarrow{\pi} \mathcal{B})$$

given by composing with π. This functor preserves colimits. This functor has a right adjoint

$$\pi_! \colon \mathsf{Cat}_{\infty/\mathcal{E}} \ \rightleftarrows\ \mathsf{Cat}_{\infty/\mathcal{B}} \colon \pi^* \ ,$$

which we refer to as *base change*, which evaluates as $\pi^* \colon (\mathcal{K} \to \mathcal{B}) \mapsto (\mathcal{E}_{|\mathcal{K}} \to \mathcal{E})$ where $\mathcal{E}_{|\mathcal{K}} := \mathcal{K} \underset{\mathcal{B}}{\times} \mathcal{E}$ is the fiber product.

Definition 5.13. A functor $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ between ∞-categories is an *exponentiable fibration* if the base change functor π_* is a left adjoint. In this case, the right adjoint

$$\pi^* \colon \mathsf{Cat}_{\infty/\mathcal{B}} \ \rightleftarrows\ \mathsf{Cat}_{\infty/\mathcal{E}} \colon \pi_*$$

is the *exponential* functor.

By adjunction, the value of this exponential functor on $\tau \to \mathcal{E}$ has the following universal property:

For each functor $\mathcal{K} \to \mathcal{B}$, there is a natural identification of the space of functors over $\mathcal{B}$

$$\mathsf{Map}_{/\mathcal{B}}\big(\mathcal{K}, \pi_*\tau\big) \ \simeq\ \mathsf{Map}_{/\mathcal{E}}\big(\mathcal{E}_{|\mathcal{K}}, \tau\big)$$

with the space of functors over $\mathcal{E}$ from the pullback.

Observation 5.14. Notice that $\mathcal{E} \to \mathcal{B}$ is an exponentiable fibration if and only if its opposite $\mathcal{E}^{\mathsf{op}} \to \mathcal{B}^{\mathsf{op}}$ is.

Remark 5.15. Not every functor is an exponentiable fibration. For instance, base change along the functor $\{0<2\} \to [2]$ carries the pushout diagram among ∞-categories over $[2]$

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \{1<2\} \arrow[d] \\ \{0<1\} \arrow[r] & {[2]} \end{tikzcd} \qquad \text{to the diagram} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{2\} \arrow[d] \\ \{0\} \arrow[r] & \{0<2\} \end{tikzcd}$$

over $\{0<2\}$ which is *not* a pushout.

The following result is an ∞-categorical version of a result of Giraud's ([Gi]), which is also the main result in [Con] of Conduché. The result articulates a sense in which Remark 5.15 demonstrates the only obstruction to exponentiability. For convenient latter application, we state one of the assertions in the result in terms of *suspension* of an ∞-category:

For $\mathcal{J}$ an ∞-category, its *suspension* is the pushout in the diagram among ∞-categories

$$\begin{tikzcd} \mathcal{J} \times \{0<2\} \arrow[r] \arrow[d] & \mathcal{J} \times [2] \arrow[d] \\ \{0<2\} \arrow[r] & \mathcal{J}^{\triangleleft\triangleright}. \end{tikzcd}$$

This construction is evidently functorial in $\mathcal{J}$. Notice the evident fully faithful functors $\mathcal{J}^{\triangleleft} \hookrightarrow \mathcal{J}^{\triangleleft\triangleright} \hookleftarrow \mathcal{J}^{\triangleright}$ from the cones.

Lemma 5.16. *The following conditions on a functor $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ between ∞-categories are equivalent.*

(1) *The functor π is an exponentiable fibration.*
(2) *The base change functor $\pi^* \colon \mathsf{Cat}_{\infty/\mathcal{B}} \to \mathsf{Cat}_{\infty/\mathcal{E}}$ preserves colimits.*
(3) *For each functor $[2] \to \mathcal{B}$, the diagram among pullbacks*

$$\begin{tikzcd} \mathcal{E}_{|\{1\}} \arrow[r] \arrow[d] & \mathcal{E}_{|\{1<2\}} \arrow[d] \\ \mathcal{E}_{|\{0<1\}} \arrow[r] & \mathcal{E}_{|[2]} \end{tikzcd}$$

is a pushout among ∞-categories.

(4) *For each functor $[2] \to \mathcal{B}$, and for each lift $\{0\} \amalg \{2\} \xrightarrow{\{e_0\}\amalg\{e_2\}} \mathcal{E}$ along π, the canonical functor from the coend*

$$\mathcal{E}_{|\{0<1\}}(e_0, -) \bigotimes_{\mathcal{E}_{|\{1\}}} \mathcal{E}_{|\{1<2\}}(-, e_2) \xrightarrow{\circ} \mathcal{E}_{|[2]}(e_0, e_2)$$

is an equivalence of spaces.

(5) *For each functor $[2] \to \mathcal{B}$, the canonical map of spaces*

$$\underset{[p]\in\boldsymbol{\Delta}^{\mathrm{op}}}{\mathsf{colim}} \ \mathsf{Map}_{/*^{\triangleleft\triangleright}}([p]^{\triangleleft\triangleright}, \mathcal{E}_{|*^{\triangleleft\triangleright}}) \xrightarrow{\circ} \mathsf{Map}_{/\{0<2\}}(\{0<2\}, \mathcal{E}_{|\{0<2\}})$$

is an equivalence. Here we have identified $[2] \simeq *^{\triangleleft\triangleright}$ as the suspension of the terminal ∞-category, and we regard each suspension $[p]^{\triangleleft\triangleright}$ as an ∞-category over $*^{\triangleleft\triangleright}$ by declaring the fiber over the left/right cone point to be the left/right cone point.

(6) *For each functor $[2] \to \mathcal{B}$, and for each lift $\{0<2\} \xrightarrow{(e_0 \xrightarrow{h} e_2)} \mathcal{E}$ along π, the ∞-category of factorizations of h through $\mathcal{E}_{|\{1\}}$ over $[2] \to \mathcal{B}$*

$$\mathsf{B}({\mathcal{E}_{|\{1\}}}^{e_0/})_{/(e_0 \xrightarrow{h} e_2)} \ \simeq \ * \ \simeq \ \mathsf{B}({\mathcal{E}_{|\{1\}}}_{/e_2})^{(e_0 \xrightarrow{h} e_2)/}$$

has contractible classifying space. Here, the two ∞-categories in the above expression agree and are the fiber of the functor $\mathsf{ev}_{\{0<2\}} \colon \mathsf{Fun}_{/\mathcal{B}}([2], \mathcal{E}) \to \mathsf{Fun}_{/\mathcal{B}}(\{0<2\}, \mathcal{E})$ over h.

Proof. By construction, the ∞-category Cat_∞ is presentable, and thereafter each over ∞-category $\mathsf{Cat}_{\infty/\mathcal{C}}$ is presentable. The equivalence of (1) and (2) follows by way of the adjoint functor theorem (Cor. 5.5.2.9 of [Lu1]), using that base-change is defined in terms of finite limits. The equivalence of (4) and (6) visibly follows from Quillen's Theorem A. The equivalence of (4) and (5) follows upon observing the map of fiber sequences among spaces

$$\begin{tikzcd} \mathcal{E}_{|\{0<1\}}(e_0, -) \bigotimes_{\mathcal{E}_{|\{1\}}} \mathcal{E}_{|\{1<2\}}(-, e_2) \arrow[r] \arrow[d, "\circ"] & \underset{[p]\in\boldsymbol{\Delta}^{\mathrm{op}}}{\mathsf{colim}} \ \mathsf{Map}_{/*^{\triangleleft\triangleright}}([p]^{\triangleleft\triangleright}, \mathcal{E}_{|*^{\triangleleft\triangleright}}) \arrow[r, "\mathsf{ev}_{0,2}"] \arrow[d, "\circ"] & \mathcal{E}_{|\{0\}} \times \mathcal{E}_{|\{2\}} \arrow[d, "="] \\ \mathcal{E}_{|[2]}(e_0, e_2) \arrow[r] & \mathsf{Map}_{/\{0<2\}}(\{0<2\}, \mathcal{E}_{|\{0<2\}}) \arrow[r, "\mathsf{ev}_{0,2}"] & \mathcal{E}_{|\{0\}} \times \mathcal{E}_{|\{2\}}, \end{tikzcd}$$

where the top sequence is indeed a fibration sequence because pullbacks are universal in the ∞-category of spaces. By construction, there is the pushout expression $\{0<1\} \underset{\{1\}}{\amalg} \{1<2\} \xrightarrow{\simeq} [2]$ in Cat_∞; this shows (2) implies (3).

We now prove the equivalence between (3) and (5). Consider an ∞-category $\mathcal{Z}$ under the diagram $\mathcal{E}_{|\{0<1\}} \leftarrow \mathcal{E}_{|\{1\}} \to \mathcal{E}_{|\{1<2\}}$. We must show that there is a unique functor $\mathcal{E}_{|[2]} \to \mathcal{Z}$ under this

diagram. To construct this functor, and show it is unique, it is enough to do so between the complete Segal spaces these ∞-categories present:

$$\mathsf{Map}([\bullet],\mathcal{E}_{|[2]}) \overset{\exists!}{\dashrightarrow} \mathsf{Map}([\bullet],\mathcal{Z})$$

under $\mathsf{Map}([\bullet],\mathcal{E}_{|\{0<1\}}) \leftarrow \mathsf{Map}([\bullet],\mathcal{E}_{|\{1\}}) \rightarrow \mathsf{Map}([\bullet],\mathcal{E}_{|\{1<2\}})$.

So consider a functor $[p] \xrightarrow{f} [2]$ between finite non-empty linearly ordered sets. Denote the linearly ordered subsets $A_i := f^{-1}(i) \subset [p]$. We have the diagram among ∞-categories

(19)
$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_1 \star A_2 \arrow[d] \\ A_0 \star A_1 \arrow[r] & {[p]} \end{tikzcd} \qquad \text{over the diagram} \qquad \begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \{1<2\} \arrow[d] \\ \{0<1\} \arrow[r] & {[2].} \end{tikzcd}$$

We obtain the solid diagram among spaces of functors

(20)
$$\begin{tikzcd}[column sep=small] \mathsf{Map}_{/\{0<1\}}(A_0 \star A_1, \mathcal{E}_{|\{0<1\}}) \arrow[ddd] & \mathsf{Map}_{/\{1\}}(A_1, \mathcal{E}_{|\{1\}}) \arrow[l] \arrow[r] \arrow[ddd, bend right] & \mathsf{Map}_{/\{1<2\}}(A_1 \star A_2, \mathcal{E}_{|\{1<2\}}) \arrow[ddd] \\ & \mathsf{Map}_{/[2]}([p], \mathcal{E}) \arrow[ul] \arrow[ur] \arrow[d, dashed, "\exists!"] & \\ & \mathsf{Map}([p], \mathcal{Z}) \arrow[dl] \arrow[dr] & \\ \mathsf{Map}_{/\{0<1\}}(A_0 \star A_1, \mathcal{Z}) & \mathsf{Map}_{/\{1\}}(A_1, \mathcal{Z}) \arrow[l] \arrow[r] & \mathsf{Map}_{/\{1<2\}}(A_1 \star A_2, \mathcal{Z}) \end{tikzcd}$$

and we wish to show there is a unique filler, as indicated.

Case that f is consecutive: In this case the left square in (19) is a pushout. It follows that the upper and the lower flattened squares in (20) are pullbacks. And so there is indeed a unique filler making the diagram (20) commute.

Case that f is not consecutive: In this case $A_1 = \emptyset$ and $A_0 \neq \emptyset \neq A_2$. Necessarily, there are linearly ordered sets B_0 and B_2 for which $B_0^{\triangleright} \simeq A_0$ and $B_2^{\triangleleft} \simeq A_2$. We recognize $B_0^{\triangleright} \coprod_{\{0\}} \{0 < 2\} \coprod_{\{2\}} B_2^{\triangleleft} \xrightarrow{\simeq} [p]$ as an iterated pushout. So the canonical maps among spaces to the iterated pullbacks

$$\mathsf{Map}_{/[2]}([p], \mathcal{E}_{|[2]}) \xrightarrow{\simeq} \mathsf{Map}(B_0^{\triangleright}, \mathcal{E}_{|\{0\}}) \underset{\mathcal{E}^{\sim}_{|\{0\}}}{\times} \mathsf{Map}_{/\{0<2\}}(\{0<2\}, \mathcal{E}_{|\{0<2\}}) \underset{\mathcal{E}^{\sim}_{|\{2\}}}{\times} \mathsf{Map}(B_2^{\triangleleft}, \mathcal{E}_{|\{2\}})$$

and

$$\mathsf{Map}([p], \mathcal{Z}) \xrightarrow{\simeq} \mathsf{Map}(B_0^{\triangleright}, \mathcal{Z}) \underset{\mathcal{Z}^{\sim}}{\times} \mathsf{Map}(\{0<2\}, \mathcal{Z}) \underset{\mathcal{Z}^{\sim}}{\times} \mathsf{Map}(B_2^{\triangleleft}, \mathcal{Z})$$

are equivalences. This reduces us to the case that $[p] \to [2]$ is the functor $\{0 < 2\} \to [2]$. We have the solid diagram among spaces

$$\begin{tikzcd} \mathsf{Map}_{/\{0<2\}}(\{0<2\}, \mathcal{E}_{|\{0<2\}}) \arrow[r, dashed, "\exists!"] & \mathsf{Map}(\{0<2\}, \mathcal{Z}) \\ {|\mathsf{Map}_{/[2]}([\bullet]^{\triangleleft\triangleright}, \mathcal{E}_{|*^{\triangleleft\triangleright}})|} \arrow[u, "\circ"] \arrow[r] & {|\mathsf{Map}([\bullet]^{\triangleleft\triangleright}, \mathcal{Z})|.} \arrow[u, "\simeq"', "\circ"] \end{tikzcd}$$

The right vertical map is an equivalence by the Yoneda lemma for ∞-categories. (Alternatively, the domain is the classifying space of the ∞-category which is the unstraightening of the indicated functor from $\mathbf{\Delta}^{\mathsf{op}}$ to spaces, and the codomain maps to this ∞-category finally.) Assumption (5) precisely gives that the left vertical map is an equivalence. The unique filler follows.

It is immediate to check that the unique fillers just constructed are functorial among finite non-empty linearly ordered sets over [2].

It remains to show (4) implies (1). To do this we make use of the presentation $\mathsf{Cat}_\infty \hookrightarrow \mathsf{PShv}(\mathbf{\Delta})$ as complete Segal spaces. Because limits and colimits are computed value-wise in $\mathsf{PShv}(\mathbf{\Delta})$, and because colimits in the ∞-category $\mathcal{S}\mathsf{paces}$ are universal, then colimits in $\mathsf{PShv}(\mathbf{\Delta})$ are universal as well. Therefore, the base change functor

$$\pi^*\colon\ \mathsf{PShv}(\mathbf{\Delta})_{/\mathcal{B}} \longrightarrow \mathsf{PShv}(\mathbf{\Delta})_{/\mathcal{E}} : \widetilde{\pi}_*$$

has a right adjoint, as notated. Because the presentation $\mathsf{Cat}_\infty \hookrightarrow \mathsf{PShv}(\mathbf{\Delta})$ preserves limits, then the functor $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ is an exponentiable fibration provided this right adjoint $\widetilde{\pi}_*$ carries complete Segal spaces over $\mathcal{E}$ to complete Segal spaces over $\mathcal{B}$.

So let $\tau \to \mathcal{E}$ be a complete Segal space over $\mathcal{E}$. To show the simplicial space $\widetilde{\pi}_*\tau$ satisfies the Segal condition we must verifty that, for each functor $[p] \to \mathcal{E}$ with $p > 0$, the canonical map of spaces of simplicial maps over $\mathcal{E}$

$$\mathsf{Map}_{/\mathcal{E}}([p], \widetilde{\pi}_*\tau) \longrightarrow \mathsf{Map}_{/\mathcal{E}}(\{0<1\}, \widetilde{\pi}_*\tau) \underset{\mathsf{Map}_{/\mathcal{E}}(\{1\}, \widetilde{\pi}_*\tau)}{\times} \mathsf{Map}_{/\mathcal{E}}(\{1<\cdots<p\}, \widetilde{\pi}_*\tau)$$

is an equivalence. Using the defining adjunction for $\widetilde{\pi}_*$, this map is an equivalence if and only if the canonical map of spaces of functors

$$\mathsf{Map}_{/\mathcal{B}}(\pi^*[p], \tau) \longrightarrow \mathsf{Map}_{/\mathcal{B}}(\pi^*\{0<1\}, \tau) \underset{\mathsf{Map}_{/\mathcal{B}}(\pi^*\{1\}, \tau)}{\times} \mathsf{Map}_{/\mathcal{B}}(\pi^*\{1<\cdots<p\}, \tau)$$

is an equivalence. This is the case provided the canonical functor among pullback ∞-categories from the pushout ∞-category

$$\mathcal{E}_{|\{0<1\}} \underset{\mathcal{E}_{|\{1\}}}{\amalg} \mathcal{E}_{|\{1<\cdots<p\}} \longrightarrow \mathcal{E}_{|[p]}$$

is an equivalence between ∞-categories over $\mathcal{B}$. (Here we used the shift in notation $\pi^*\mathcal{K} := \mathcal{E}_{|\mathcal{K}}$ for each functor $\mathcal{K} \to \mathcal{B}$.) This functor is clearly essentially surjective, so it remains to show this functor is fully faithful. Let e_i and e_j be objects of $\mathcal{E}$, each which lies over the object of $[p]$ indicated by the subscript. We must show that the map between spaces of morphisms

$$\big(\mathcal{E}_{|\{0<1\}} \underset{\mathcal{E}_{|\{1\}}}{\amalg} \mathcal{E}_{|\{1<\cdots<p\}}\big)(e_i, e_j) \longrightarrow \mathcal{E}_{|[p]}(e_i, e_j)$$

is an equivalence. This is directly the case whenever $1 < i \leq j \leq p$ or $0 \leq i \leq j \leq 1$. We are reduced to the case $i = 0 < j$. This map is identified with the map from the coend

$$\mathcal{E}_{|\{0<1\}}(e_0, -) \bigotimes_{\mathcal{E}_{|\{1\}}} \mathcal{E}_{|\{1<j\}}(-, e_j) \xrightarrow{\circ} \mathcal{E}_{|\{0<1<j\}}(e_0, e_j)\ .$$

Condition (4) exactly gives that this map is an equivalence, as desired.

It remains to verify this Segal space $\widetilde{\pi}_*\tau$ satisfies the univalence condition. So consider a univalence diagram $\mathcal{U}^{\triangleright} \to \mathcal{B}$. We must show that the canonical map

$$\mathsf{Map}_{/\mathcal{B}}(*, \widetilde{\pi}_*\tau) \longrightarrow \mathsf{Map}_{/\mathcal{B}}(\mathcal{U}, \widetilde{\pi}_*\tau)$$

is an equivalence of spaces of maps between simplicial spaces over $\mathcal{B}$. Using the defining adjunction for $\widetilde{\pi}_*$, this map is an equivalence if and only if the map of spaces

$$\mathsf{Map}_{/\mathcal{E}}(\mathcal{E}_{|*}, \tau) \longrightarrow \mathsf{Map}_{/\mathcal{B}}(\mathcal{E}_{|\mathcal{U}}, \tau)$$

is an equivalence. Because the presentation of $\mathcal{B}$ as a simplicial space is complete, there is a canonical equivalence $\mathcal{E}_{|\mathcal{U}} \simeq \mathcal{E}_{|*} \times \mathcal{U}$ over $\mathcal{U}$. That the above map is an equivalence follows because the presentation of τ as a simplicial space is complete.

□

Corollary 5.17. *If a functor $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ is either a coCartesian fibration or a Cartesian fibration then it is an exponentiable fibration. In particular, both left fibrations and right fibrations are exponentiable fibrations.*

Proof. Using Observation 5.14, each case implies the other. So we will concern ourselves only with the coCartesian case. We will invoke criterion (6) of Lemma 5.16. So fix a functor $[2] \to \mathcal{B}$. For each $0 \leq i < j \leq 2$ we will denote $f^{ij}\colon \{i<j\} \to \mathcal{B}$ for the resulting morphisms of $\mathcal{B}$; we will denote $f_!^{ij}\colon \mathcal{E}_{|\{i\}} \to \mathcal{E}_{|\{j\}}$ for the coCartesian functor between fiber ∞-categories; and we will denote u^{ij} for a π-coCartesian lift of f^{ij}. Because $f^{12} \circ f^{01} \simeq f^{02}$ then $f_!^{12} \circ f_!^{01} \simeq f_!^{12}$ and $u^{12} \circ u^{01} \simeq u^{02}$, whenever the latter composition has meaning.

Consider a lift $\{0 < 2\} \xrightarrow{(e_0 \xrightarrow{h} e_2)} \mathcal{E}$. There is a unique factorization $h\colon e_0 \xrightarrow{u^{01}} f_!^{01}(e_0) \xrightarrow{u^{12}} f_!^{02}(e_0) \xrightarrow{\overline{h}} e_2$ in which $\overline{h}$ is a morphism in the fiber ∞-category $\mathcal{E}_{|\{2\}}$. We have the diagram in $\mathcal{E}$

$$\begin{tikzcd}
e_0 \arrow[rr, "h"] \arrow[dr, "u^{01}"'] & & e_2 \\
& f_!^{01}(e_0) \arrow[ur, "\overline{h}\circ u^{12}"'] &
\end{tikzcd} ,$$

which is a lift of $[2] \to \mathcal{B}$ along π. As so, it depicts an object of the ∞-category $(\mathcal{E}_{|\{1\}}{}^{e_0/})_{/(e_0 \xrightarrow{h} e_2)}$. Direct from universal properties of π-coCartesian morphisms, this object is initial. Therefore, the classifying space $\mathsf{B}(\mathcal{E}_{|\{1\}}{}^{e_0/})_{/(e_0 \xrightarrow{h} e_2)}$ is contractible.

□

5.3.2. **Doubly based finite sets.** Recall from the top of §3.5 the ∞-category $\mathsf{Fin}_{*\star}$ over Fin_*. The defining functor $\mathsf{Fin}_{*\star} \hookrightarrow \mathsf{Fin}_*^{\star+/}$ is a monomorphism. That is, for each functor $\mathcal{K} \to \mathsf{Fin}_*$, the map

$$\mathsf{Map}_{/\,\mathsf{Fin}_*}(\mathcal{K}, \mathsf{Fin}_{*\star}) \hookrightarrow \mathsf{Map}_{/\,\mathsf{Fin}_*}(\mathcal{K}, \mathsf{Fin}_*^{\star+/}) \tag{21}$$

is a monomorphism of spaces. The given functor $\mathcal{K} \to \mathsf{Fin}_*$ classifies a left fibration $\mathcal{E} \to \mathcal{K}$, equipped with a section $\sigma_+\colon \mathcal{K} \to \mathcal{E}$, with the property that, for each $k \in \mathcal{K}$, the fiber $\mathcal{E}_k$ is a 0-type. As so, the monomorphism of spaces (21) is naturally identified as the inclusion of path components

$$\{\mathcal{K} \xrightarrow{\sigma} \mathcal{E} \mid \text{ for each } k \in \mathcal{K},\ \sigma_+(k) \nsim \sigma(k) \in \mathcal{E}\} \hookrightarrow \mathsf{Map}_{/\mathcal{K}}(\mathcal{K}, \mathcal{E})$$

consisting of those sections which are object-wise never equivalent to the given section σ_+. In the case that $\mathcal{K}$ is a suspension, this monomorphism (21) is yet more explicit, which we highlight as the following simple result of this discussion.

Observation 5.18. Let $\mathcal{J}^{\triangleleft\triangleright} \to \mathsf{Fin}_*$ be a functor from a suspension. Denote the value of this functor on the left/right cone point as the based finite set L_+/R_+, and denote the value of this functor on the unique morphism between cone points as the based map $L_+ \xrightarrow{f_0} R_+$. Evaluation at the left cone point defines an identification of the monomorphism of spaces of sections $\mathsf{Map}_{/\,\mathsf{Fin}_*}(\mathcal{J}^{\triangleleft\triangleright}, \mathsf{Fin}_{*\star}) \hookrightarrow \mathsf{Map}_{/\,\mathsf{Fin}_*}(\mathcal{J}^{\triangleleft\triangleright}, \mathsf{Fin}_*^{\star+/})$ with the inclusion of sets

$$f_0^{-1}(R) \hookrightarrow L_+ .$$

Lemma 5.19. *The projection $\mathsf{Fin}_{*\star} \to \mathsf{Fin}_*$ is an exponentiable fibration.*

Proof. We verify (4) of Lemma 5.16 applied to the functor $\mathsf{Fin}_{*\star} \to \mathsf{Fin}_*$. Let $[2] \xrightarrow{I_+ \xrightarrow{f} J_+ \xrightarrow{g} K_+} \mathsf{Fin}_*$ be a functor. Choose lifts (I_+, i) and (K_+, k) to $\mathsf{Fin}_{*\star}$. Observation 5.18 identifies the the space of morphisms $\mathsf{Fin}_{*\star|[2]}\big((I_+, i), (K_+, k)\big)$ as terminal if $g(f(i)) = k$ and as empty otherwise. In the latter case, the coend too is empty, which verifies the criterion. So assume $g(f(i)) = k$. We must show that the coend too is terminal. This coend is indexed by a 0-type, so it is identified simply as the space

$$\coprod_{j \in J} \mathsf{Fin}_{*\star|\{0<1\}}\big((I_+, i), (J_+, j)\big) \times \mathsf{Fin}_{*\star|\{1<2\}}\big((J_+, j), (K_+, k)\big) .$$

Using Observation 5.18 we see that the cofactor indexed by $f(i) \in J$ is terminal while the other cofactors are empty.

$\square$

5.3.3. **Absolute exit-paths.** We prove that the functor $\mathcal{E}\mathsf{xit} \to \mathcal{B}\mathsf{un}$ between ∞-categories, given by forgetting section data, is exponentiable. This result is essential for our method of defining structured versions of $\mathcal{B}\mathsf{un}$.

Lemma 5.20. *The functor* $\mathcal{E}\mathsf{xit} \to \mathcal{B}\mathsf{un}$ *is an exponentiable fibration.*

Proof. We check the criterion (5) of Lemma 5.16. Fix a functor $[2] \to \mathcal{B}\mathsf{un}$. This functor classifies a constructible bundle $X \to \Delta^2$. We will denote the restriction $X_S := X_{|\Delta^S}$ for each non-empty linearly ordered subset $S \subset \{0 < 1 < 2\}$.

Through §3 of [AFR] we establish a natural identification of the ∞-category of lifts

$$\mathsf{Fun}_{/\{0<2\}}(\{0<2\}, \mathcal{E}\mathsf{xit}_{|\{0<2\}}) \;\simeq\; \mathsf{Exit}\big(\mathsf{Link}_{X_0}(X_{02})\big)$$

as the exit-path ∞-category of the link. Likewise, there is an identification of the ∞-category of lifts

$$\mathsf{Fun}_{/[2]}([2], \mathcal{E}\mathsf{xit}_{|[2]}) \;\simeq\; \mathsf{Exit}\Big(\mathsf{Link}_{\mathsf{Link}_{X_0}(X_{01})}\big(\mathsf{Link}_{X_0}(X)\big)\Big)$$

as the exit-path ∞-category of the iterated link. In §6 of [AFR] we construct a refinement morphism among stratified spaces

$$\gamma\colon\; \mathsf{Link}_{\mathsf{Link}_{X_0}(X_{01})}\big(\mathsf{Link}_{X_0}(X)\big) \;\longrightarrow\; \mathsf{Link}_{X_0}(X_{02})$$

from the iterated link to the link – in brief, this morphism is obtained using flows of collaring vector fields of links inside blow-ups. This morphism is compatible with the above identifications in that there is a commutative diagram among ∞-categories

$$\begin{tikzcd}
\mathsf{Fun}_{/[2]}([2], \mathcal{E}\mathsf{xit}_{|[2]}) \arrow[r, "\circ", "\mathsf{ev}_{\{0<2\}}"'] \arrow[d, "\simeq"] & \mathsf{Fun}_{/\{0<2\}}(\{0<2\}, \mathcal{E}\mathsf{xit}_{|\{0<2\}}) \arrow[d, "\simeq"] \\
\mathsf{Exit}\Big(\mathsf{Link}_{\mathsf{Link}_{X_0}(X_{01})}\big(\mathsf{Link}_{X_0}(X)\big)\Big) \arrow[r, "\mathsf{Exit}(\gamma)"] & \mathsf{Exit}\big(\mathsf{Link}_{X_0}(X_{02})\big).
\end{tikzcd}$$

Restricting the righthand terms of this diagram to maximal ∞-subgroupoids gives the commutative diagram among ∞-categories

$$\begin{tikzcd}
\mathsf{Fun}_{/[2]}([2], \mathcal{E}\mathsf{xit}_{|[2]})_{|\,\mathsf{Map}_{/\{0<2\}}(\{0<2\}, \mathcal{E}\mathsf{xit}_{|\{0<2\}})} \arrow[r, "\circ", "\mathsf{ev}_{\{0<2\}}"'] \arrow[d, "\simeq"] & \mathsf{Map}_{/\{0<2\}}(\{0<2\}, \mathcal{E}\mathsf{xit}_{|\{0<2\}}) \arrow[d, "\simeq"] \\
\coprod_{p\in P}\Big(\mathsf{Link}_{\mathsf{Link}_{X_0}(X_{01})}\big(\mathsf{Link}_{X_0}(X)\big)\Big)_{|\,\mathsf{Link}_{X_0}(X_{02})_p} \arrow[r, "\mathsf{Exit}(\gamma)_|"] & \coprod_{p\in P}\mathsf{Link}_{X_0}(X_{02})_p
\end{tikzcd}$$

where the coproducts are indexed by the strata of the link.

To verify criterion (5) of Lemma 5.16 we must explain why the top horizontal functor induces an equivalence on classifying spaces. By the commutativity of the previous recent diagram, we must explain why the bottom horizontal map induces an equivalence on classifying spaces. In §4 of [AFT] it is proved that, for $\widetilde{X} \to X$ a refinement between stratified spaces, the associated functor between exit-path ∞-categories $\mathsf{Exit}(\widetilde{X}) \to \mathsf{Exit}(X)$ is a localization. In particular, for each stratum $X_p \subset X$, the map $\mathsf{Exit}(\widetilde{X}_{|X_p}) \to X_p$ induces an equivalence on classifying spaces.

$\square$

5.4. **Higher idempotents.** In this section we define the notion of a k-idempotent in an (∞, n)-category, and more generally in a Segal $\mathbf{\Theta}_n$-space. We show that, should a Segal $\mathbf{\Theta}_n$-space have only identity k-idempotents for each $0 < k \leq n$, then it presents an (∞, n)-category.

Let Idem denote the unique two-element monoid which is not a group. This monoid is commutative, and so defines a functor $\mathsf{Idem}\colon \mathsf{Fin}_* \to \mathsf{Set} \hookrightarrow \mathcal{S}\mathsf{paces}$ which satisfies a Segal condition in the sense of [Se1]. Precomposing with the simplicial circle defines a functor

$$\mathfrak{B}\mathsf{Idem}\colon \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{c_1/\partial c_1} \mathsf{Fin}_* \xrightarrow{\mathsf{Idem}} \mathcal{S}\mathsf{paces} ~.$$

For each $0 < k \leq n$ consider the presheaf Idem^k on $\mathbf{\Theta}_n$ which is the left Kan extension:

$$\begin{tikzcd}
\mathbf{\Delta}^{\mathsf{op}} \arrow[r, "\mathfrak{B}\mathsf{Idem}"] \arrow[d, "c_{k-1}\wr -"'] & \mathcal{S}\mathsf{paces} \\
\mathbf{\Theta}_k^{\mathsf{op}} \arrow[r, "\iota_k"] & \mathbf{\Theta}_n^{\mathsf{op}} ~. \arrow[u, dashed, "\mathsf{Idem}^k"']
\end{tikzcd}$$

Because $\mathsf{Idem}\colon \mathsf{Fin}_* \to \mathcal{S}\mathsf{paces}$ is a Segal Fin_*-space, then this $\mathbf{\Theta}_n$-space Idem^k too is Segal. Because the maximal subgroup of the monoid Idem is trivial, then this Segal $\mathbf{\Theta}_n$-space Idem^k is univalent. Therefore, Idem^k presents an (∞, n)-category. As so, Idem^k corepresents a k-idempotent of a Segal $\mathbf{\Theta}_n$-space.

Observation 5.21. For each $0 < k \leq n$, the projection $\mathsf{Idem}^k \to c_{k-1}$ between Segal $\mathbf{\Theta}_n$-spaces is an epimorphism. This follows by induction, using that the map of monoids $\mathsf{Idem} \to *$ is an epimorphism.

We say a k-idempotent of a Segal $\mathbf{\Theta}_n$-space, $\mathsf{Idem}^k \to \mathcal{C}$, is a *$k$-identity* if there is a factorization $\mathsf{Idem}^k \to c_{k-1} \to \mathcal{C}$. After Observation 5.21, it is a *condition* on a k-idempotent to be an identity.

Lemma 5.22. *Let $\mathcal{C}$ be a Segal $\mathbf{\Theta}_n$-space. Suppose, for each $0 < k \leq n$, that each k-idempotent of $\mathcal{C}$ is an identity, by which we mean each map of presheaves $\mathsf{Idem}^k \to \mathcal{C}$ factors through $\mathsf{Idem}^k \to c_{k-1}$. Then $\mathcal{C}$ presents an (∞, n)-category.*

Proof. We will make ongoing use of Observation 3.43 where a number of monomorphisms among $\mathbf{\Theta}_i$'s are established. Consider a univalence diagram $\mathcal{E}^{\triangleright} \to \mathbf{\Theta}_n$. From its definition, there is a maximal $0 < k \leq n$ for which there is a factorization

$$\mathcal{E}^{\triangleright} \longrightarrow \mathbf{\Delta} \xrightarrow{\iota} \mathbf{\Theta}_{n-k+1} \xrightarrow{c_{k-1}\wr -} \mathbf{\Theta}_n$$

through a univalence diagram of $\mathbf{\Delta}$. In this way, it is enough to consider the case $n = k = 1$, where the $\mathcal{E} \to \mathbf{\Theta}_n$ is the diagram in $\mathbf{\Delta}$

$$* \leftarrow \{0 < 2\} \to \{0 < 1 < 2 < 3\} \leftarrow \{1 < 3\} \to * ~.$$

By definition, the colimit $\widetilde{E}$ of this diagram in the ∞-category $\mathsf{PShv}^{\mathsf{Segal}}(\mathbf{\Delta})$, of Segal $\mathbf{\Delta}$-spaces, corepresents an equivalence. This diagram receives an evident map from the diagram in $\mathbf{\Delta}$

$$* \leftarrow \{0 < 2\} \to \{0 < 1 < 2\} ~.$$

The colimit $\widetilde{R}$ of this latter diagram in Segal $\mathbf{\Delta}$-spaces corepresents a pair of retractions. By direct inspection, this Segal $\mathbf{\Delta}$-space $\widetilde{R}$ is complete, and therefore presents an ∞-category. We obtain a comosite map of Segal $\mathbf{\Delta}$-spaces

$$\mathsf{Idem}^1 \longrightarrow \widetilde{R} \longrightarrow \widetilde{E} ~.$$

Therefore, there is a natural map $\mathsf{Map}(\widetilde{E}, \mathcal{C}) \longrightarrow \mathsf{Map}(\mathsf{Idem}^1, \mathcal{C})$ from the space of equivalences in $\mathcal{C}$ to the space of pairs of idempotents of $\mathcal{C}$. This map fits into a commutative triangle of maps among spaces

$$\begin{tikzcd}
\mathsf{Map}(\widetilde{E}, \mathcal{C}) \arrow[rr] & & \mathsf{Map}(\mathsf{Idem}^1, \mathcal{C}) \\
& \mathsf{Map}(c_0, \mathcal{C}) \simeq \mathcal{C}[0] \arrow[ul] \arrow[ur] &
\end{tikzcd}
~.$$

It is standard that the leftward diagonal map is a monomorphism. The assumption on $\mathcal{C}$ exactly grants that the rightward diagonal map is an equivalence. It follows that this diagram is comprised of equivalences among spaces. This proves that $\mathcal{C}$ carries the univalent diagram to a limit diagram, as desired.

□

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